SOME OBSERVATIONS ON UNDERGRADUATE MATHEMATICS IN AMERICAN COLLEGES AND UNIVERSITIES

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1. Introduction. During the academic year 1951–1952 it was my privilege, under a grant from the Ford Foundation, to visit the mathematics departments of thirty-three American colleges and universities. These institutions, located in the East, the Midwest, and the Far West, extended from Boston to Los Angeles, from Philadelphia to Seattle. Among them were privately endowed universities, state universities, private colleges, and city colleges. They were selected on the basis of academic excellence, and it is believed that they represent institutions of the highest scholastic standing in the regions mentioned. My primary mission was to study undergraduate mathematics programs. The members of the departments visited were most helpful and generous with their time in supplying information about their own institutions and in participating in discussions on various aspects of undergraduate mathematics instruction. This experience has resulted in certain impressions concerning the present state of undergraduate mathematical education in this country, at least as reflected by the institutions visited, which may be worth passing on to others interested in this matter.

2. Mathematics in general education. Relative to the status of mathematics as a required subject, in only four of the institutions visited is it required of all students for graduation. In three others, mathematics or philosophy is required. The most usual general requirement is one which specifies that a certain number of courses be selected from a group consisting of the natural sciences and mathematics. A question of interest here is the extent to which mathematics is regarded as an essential subject in a liberal education. Concerning this point, I made the following observations. In the institutions which have inaugurated "general education" courses in the humanities, the social sciences, or the natural sciences, with but a single exception mathematics has no place in any of these courses. Indeed, in some institutions which have had the longest experience with such courses, and also in other institutions, the mathematical needs of only those students who follow certain special curricula are seriously considered. This suggests that some mathematics departments are not especially concerned with the contribution their subject might make to a liberal education or else feel that this contribution is automatically provided by the usual introductory courses. On the other hand, one finds individual members of these departments vitally interested in this question, but who, for various reasons, are unable at present to implement their convictions with changes in course content or organization. Another interesting observation is that, generally speaking, in the West one finds fewer instances of mathematics considered as a basic part of a liberal arts program than in the East.

3. Types of freshman courses. The traditional college algebra and trigonometry are still the usual freshman courses in most institutions. However, eighteen of the institutions visited provide courses of a non-traditional character which at least some students can elect or can take to satisfy requirements involving mathematics. These non-traditional courses are usually designed for students not planning to pursue work in which mathematics is needed as a tool. Their content frequently includes elements of analytic geometry and calculus as well as some algebra and trigonometry, and occasionally topics from even more advanced subjects such as number theory and topology. Recently, in some places probability and statistics have come to be regarded as appropriate subjects for inclusion in such a course. With the increasing application of mathematical statistics to more and more areas of human knowledge, there seem to be cogent reasons for teaching as many students as possible something about the nature of statistical inference. In some courses considerable attention is devoted to logic and the character of mathematics as a logical structure. Most of the people with whom I talked agree that much of the trigonometry and some of the algebra in the traditional freshman course can well be replaced by mathematics that is more interesting and of greater significance. Especially is this true for the student who will not take any more mathematics.

The type of course which offers most promise of substantial contribution to a general education is not adequately described by merely listing the topics covered. The spirit in which the subject is treated is of the greatest importance. An understanding of the nature and significance of mathematics is sought through an emphasis on basic concepts, the logical processes used in developing the subject, and the relation of the discipline to other fields through a consideration of its origins and its applications. Techniques, of course, are necessary, but there is plenty of evidence that many students pass their freshman courses by memorizing techniques without obtaining the slightest insight into the true nature of mathematics. Such a procedure could hardly contribute much to a liberal education. These courses, whose nature is so sketchily suggested here, have as their primary objective the fuller realization of the educational values long believed to be inherent in the discipline of mathematics. It is to be emphasized that it is not thought that this can be accomplished by use of descriptive material *about* the subject. Serious mathematics must be the backbone of the course. But exactly what topics constitute the most suitable content for such a course is by no means fully decided. Much experimentation remains to be done. Increasing dissatisfaction with the inadequate contributions of traditional courses to a general education is impelling some institutions to undertake serious investigation in this area. The number of good textbooks suitable for a course of this character is extremely small. This fact has undoubtedly discouraged some institutions from instituting such a course. It seems likely that in time this deterrent will be eliminated.

It would appear that one of the characteristics of many mathematics teachers is their reluctance to try something new. As a result they frequently get

into a rut—it is easy to do in elementary courses in this field—and the listlessness of their students is a reflection of their own boredom with a subject grown stale from endless repetition. The institutions in which the quality of instruction impresses one most favorably are those in which the courses are constantly studied for possible improvement. A stabilized course tends to become a stagnant course.

The freshman course designed for the non-specialist, which we have been discussing, is thought by some also to be the best type of introductory course for students who will take further courses in the field. It is recognized that in the transition from courses of this type to advanced work provision must be made to supply certain knowledge and skills not included in a course planned as terminal.

4. Analytic geometry and calculus. In some institutions with high entrance requirements the usual freshman course consists of analytic geometry and calculus. Whether taught in the freshman year or later, there is an increasing tendency to teach these two subjects together. A goodly number of mathematicians are convinced that from a purely mathematical viewpoint there is much to be gained by teaching them together. Also, from the standpoint of the student's whole program an earlier introduction to the calculus has many advantages; one obvious one is the availability of this tool for use in elementary physics courses. There are topics in analytic geometry which, while interesting mathematics in themselves, are not a necessary part in the mainstream development, and consequently can be and frequently are omitted in these combination courses. Several institutions are experimenting with a more rigorous type of calculus course for their better students. There is considerable difference of opinion as to the degree of rigor feasible in a first course in calculus. Here the importance of variation in ability among students becomes very evident. It appears that only the best ones are able at this stage to assimilate the type of rigor proposed. Another important factor may be the nature of previous work in mathematics, whether a certain maturity in understanding mathematical concepts and proofs has been developed.

5. Upper college courses. In the work of the junior and senior years, a fairly recent innovation is the offering at the majority of the institutions visited of one or more courses in modern algebra. The proper content and level of abstractness for these courses are by no means universally agreed upon. There are those who believe that the first course should be fairly concrete, perhaps devoted mostly to matrices and vector spaces, while others think an introduction to various abstract algebraic systems is the most valuable type of first course. In any event, in the near future instructors will have a considerably larger number of textbooks from which to choose than has been the case previously. In many institutions courses in classical theory of equations are being replaced by some form of modern algebra.

In analysis there is considerable thought being given to courses which make

the transition from elementary calculus to graduate courses in function theory. The nature of these courses varies rather widely among different institutions.

One also finds great variation among institutions in interest and course offering in geometry. Occasionally, mention is made of the need for the reformulation of courses in this field. One possible future trend is the teaching of certain parts of geometry and algebra together—such as introductory projective geometry and linear algebra.

Most departments now offer one or several courses in mathematical statistics. A few institutions provide sequences of special courses designed to give the necessary mathematical training for social scientists. The increasing interest in applications of mathematics in the social sciences may influence courses in the calculus, matrix theory, and other topics in algebra. It is rather surprising to learn that a knowledge of the structure of some of the abstract algebraic systems is proving useful in certain types of investigations in the social science field.

In about half of the institutions visited, something besides formal courses is provided for undergraduates specializing in mathematics. These extra activities take the form of reading courses, honors work, seminars, tutorials, *etc.* The main purpose is to have the student do some independent work under appropriate supervision. The chief deterrent to a more extensive occurrence of these practices is the cost in terms of faculty time. In six of the institutions comprehensive examinations in mathematics are given to all seniors specializing in the subject, and in three others these examinations are given students in the honors program. At several places, theses in the major subject are required of seniors.

6. Teacher training. In the realm of teacher training, courses in algebra and geometry aimed at meeting the special needs of secondary school teachers are frequently offered. It would appear that the character of these courses in some institutions should be scrutinized for their relevance to their purported purpose. Occasionally, courses in fundamental concepts and the history of mathematics are recommended for prospective teachers. It must be admitted that university mathematics departments do not always fully discharge their obligations in the training of teachers. Too often mathematicians are content to complain of the poor preparation students receive in high schools without taking any steps to determine how departments of mathematics might help to remedy the situation. As a result, too large a part of the training of high school teachers is frequently left to less competent agencies.

7. Universities versus colleges. It is generally conceded that undergraduate education in the large universities is by and large not up to the high standards set by the good small colleges. There are, of course, many reasons for this: differences in admission requirements, extensive use in universities of graduate assistants to teach elementary courses, the presence or absence of an atmosphere conducive to good academic work, *etc.* One of the most important factors is that in universities the men with the imagination, the energy, and the enthu-

siasm to raise teaching to truly inspirational levels are frequently so heavily engaged in research, training graduate students, and other activities that they simply do not have the time to devote to elementary teaching. Thus, in many of our great universities the quality of undergraduate instruction has lagged behind that of research and graduate training. There appears to be in some places a genuine need for more mathematicians with the qualities mentioned above, drawing on their learning and inventiveness, to contribute ideas, constructive suggestions, and a part of their time to the essential task of undergraduate education.

State universities have their own peculiar problems. Many of them are due to a heterogeneous student body, representing an incredibly wide range of ability, preparation, and interest. In several states all graduates of accredited high schools must be admitted to the state university. In most of the state institutions some provision for individual differences is made in the freshman year by offering algebra courses at various levels. It is quite usual to find "intermediate algebra"—second-year high school algebra—taught in universities, and frequently college credit is given for the course. In state universities with relatively unselected freshman classes, differentiated programs of study appear to be the only feasible way of providing the type of education appropriate to various levels of ability. The present practice of setting standards and adjusting levels of teaching for the median student frequently results in something which is beyond the grasp of the poorer students and at the same time fails to challenge the better ones. This problem is recognized at many institutions but it is far from being solved.

8. Exchange of information. It became obvious to me that there is a genuine need for a freer exchange of information between institutions, particularly information regarding innovations and experiments. Perhaps the Association, through its meetings and through this MONTHLY, can make a greater contribution to this end. Also, personal correspondence and private conversation, where opportunity permits, would certainly be of great mutual benefit to everyone engaged in this process of trying to improve undergraduate mathematical education.

TYPES OF FUNCTIONS*

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1. Peculiar functions. Let f be a function whose domain of definition X is a neighborhood space. Let P be a point property of the function f. The function f is said to be *peculiar with respect to the property P* if there exists a partition of X into two subsets X_1 and X_2 each everywhere dense in X and such that the property P holds at every point of X_1 and fails to hold at every point of X_2.

For example, a function whose domain of definition is the interval (a, b), whose points of continuity are everywhere dense in (a, b) and whose points of discontinuity are everywhere dense in (a, b) is peculiar with respect to the property of continuity, and also with respect to the property of discontinuity. The function $f(x)$, which is defined in the open interval $(0, 1)$ as follows:

$$f(x) = 0 \quad \text{if} \quad x \text{ is irrational}$$
$$= 1/q \quad \text{if} \quad x = p/q, \text{ where } p \text{ and } q \text{ are relatively prime positive integers}$$

is such a peculiar function.

We give another example of a function which is peculiar with respect to continuity. In the last example the points of continuity constituted a set of measure one, while the points of discontinuity formed a set of measure zero. In the next example these characteristics are reversed for the corresponding sets, that is, the points of continuity will form a set of measure zero, while the points of discontinuity will constitute a set of measure one.

By the well known method used for the construction of the Cantor ternary set we construct a sequence of sets $N_1, N_2, N_3, \cdots, N_n, \cdots$, such that each of these sets is nowhere dense in the closed interval $[0, 1]$. The complement in $[0, 1]$ of the set N_n consists of the points of a denumerable number of non-overlapping, non-abutting open intervals. We construct the set $N_n (n = 1, 2, 3, \cdots)$ so that the sum of the lengths of the complementary intervals is $1/2^n$. Then the set S which is given by

$$S = \sum_{i=1}^{\infty} N_i$$

is a set of Baire's first category whose measure is one. Its complement is everywhere dense in $[0, 1]$ and has measure zero. We now define a sequence $\{f_n(x)\}$ of functions as follows:

$$f_n(x) = 1/2^n \quad \text{if } x \text{ is an element of } N_n,$$
$$= 0 \quad \text{if } x \text{ is not an element of } N_n.$$

* An excerpt from an address presented to the Minnesota Section of the Mathematical Association of America at the invitation of the Executive Committee on May 10, 1952.

The function

$$F(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (0 < x < 1)$$

is the required function.

We first show that $F(x)$ is discontinuous at each point of S. Let x be an element of S. Then x is an element of at least one of the sets $N_n (n = 1, 2, 3, \cdots)$. Suppose $x \in N_k$. Then $F(x) \geqq 1/2^k$. But in every neighborhood of x there are points of the everywhere dense complement of S where $F(x) = 0$. Hence $F(x)$ is discontinuous at x.

Next we show that $F(x)$ is continuous at every point of the complement of S. Let c be a point of the complement of S, and let a positive number ϵ be given. Since the series $\sum f_n(x)$ converges uniformly in $[0, 1]$ there exists a positive integer m such that the remainder

$$\sum_{n=m+1}^{\infty} f_n(x)$$

is less than ϵ for all x in $[0, 1]$. We can find a subinterval (α, β) of $[0, 1]$ containing c and such that no point of the finite set of nowhere dense sets N_1, $N_2, \cdots, N_m$ lies in (α, β). Then for every x of (α, β), x is either an element of S, or x belongs only to those nowhere dense sets N_n for which $n > m$. In either case $|F(c) - F(x)| = |F(x)| < \epsilon$, and $F(x)$ is continuous at c.

We have thus given two extreme examples of functions which are peculiar with respect to continuity.

We might mention that there can exist no function which is continuous at every rational point and discontinuous at every irrational point. This follows from the fact that the set of irrational points is not of Baire's first category while the points of discontinuity of every pointwise discontinuous function constitute a set of Baire's first category.

It is easy to construct functions which are peculiar with respect to differentiability.

We shall next give examples of functions which are peculiar with respect to a certain property which can be considered as a generalization of the notion of continuity. After that we shall show that there exist point properties with respect to which there exist no peculiar functions.

2. Neighborly functions. Let f be a function whose domain of definition is a neighborhood space X with neighborhoods denoted by N, or by N_x, and let the range of f be a neighborhood space whose neighborhoods will be denoted by M_y or by $M_{f(x)}$. The function f is said to be ***neighborly**** at a point c of its domain of definition if for every neighborhood N_c of c and for every neighborhood

* This concept was defined for the case of functions of a real variable whose ranges are metric spaces by Woodrow W. Bledsoe. Proc. Amer. Math. Soc., vol. 3, pp. 114–115, 1952.

$M_{f(c)}$ of $f(c)$ there exists a neighborhood N contained in N_c (but not necessarily containing c) such that for every element of x of N it is true that $f(x)$ is an element of $M_{f(c)}$. A function which is neighborly at each point of its domain of definition is called a ***neighborly function.*** A function is said to be ***non-neighborly*** at a point if it is not neighborly at that point.

It is obvious that if a function is continuous at a point then it is neighborly at that point.

The function $f(x)=\sin 1/x$ if $x\neq 0$, $f(0)=c$ $(-1\leqq c\leqq 1)$ is neighborly at $x=0$, but it is discontinuous at that point.

A function $f(x)$ is said to be ***pointwise discontinuous*** in X if the set of points x where $f(x)$ is continuous is everywhere dense in X, but not closed relative to X. A function $f(x)$ is ***pointwise continuous*** in X if the sets of points x where $f(x)$ is discontinuous is everywhere dense in X but not closed relative to X.

A function $f(x)$ is said to be ***pointwise non-neighborly*** in X if the set of points x where $f(x)$ is neighborly is everywhere dense in X, but not closed relative to X. A function $f(x)$ is said to be ***pointwise neighborly*** in X if the set of points x where $f(x)$ is non-neighborly is everywhere dense in X, but not closed relative to X.

The functions given in the examples of section 1 are pointwise continuous and pointwise discontinuous and also pointwise neighborly and pointwise non-neighborly in their domains of definition. Those functions are thus peculiar with respect to continuity and also with respect to neighborliness.

3. Cliquish functions.† Let the domain of definition of a function f be a neighborhood space whose neighborhoods are denoted by N or N_x, and let the range of the function be a metric space with metric ρ. The function f is said to be *cliquish at a point* c of the closure of the domain of definition if for every positive number ϵ and for every neighborhood N_c of c there exists a neighborhood N contained in N_c (but not necessarily containing c) such that for every two elements x_1 and x_2 of N it is true that

$$\rho[f(x_1), f(x_2)] < \epsilon.$$

It is obvious that if a function is continuous or neighborly at a point it is cliquish at that point. The function $f(x)$ given by the equations

$$\begin{aligned} f(x) &= \sin 1/x && \text{if } x \neq 0, \\ &= 2 && \text{if } x = 0, \end{aligned}$$

is cliquish at $x=0$, but it is not neighborly at that point.

A function which is cliquish at every point of its domain of definition is called a ***cliquish function.*** The functions of the examples in section 1 are only pointwise continuous and pointwise neighborly but they are cliquish. As a matter of fact, we shall see that every function which is pointwise non-neighborly is cliquish.

† A similar concept, called *neighborly'*, was defined by Bledsoe (loc. cit., p. 115) for functions of a real variable whose ranges are metric spaces.

A function is said to be *non-cliquish at a point* if it is not cliquish at that point. The function which is zero at each rational point and one at each irrational point is an example of a function which is non-cliquish at every point.

A function is said to be *pointwise non-cliquish* in X if the set of points where the function is cliquish is everywhere dense in X but not closed relative to X. A function is *pointwise cliquish* in X if the set of points where the functions is non-cliquish is everywhere dense in X but is not closed relative to X.

A function which was both pointwise cliquish and also pointwise non-cliquish in its domain of definition would be peculiar with respect to cliquishness. We shall now prove that there exist no such functions. Even though cliquishness of a function is similar to continuity and neighborliness of a function we have the interesting result that there exist no functions which are peculiar with respect to cliquishness.

THEOREM I. *Let f be a function with domain of definition X. If f is cliquish at each point of a set dense in X, then f is cliquish on X. In other words, there are no pointwise non-cliquish functions.*

Proof. Let f be the function given in the statement of the theorem. There exists a set C which is everywhere dense in X, and which is such that for each point c of C the function f is cliquish at c. Let x be a given point of X, and let N_x be an arbitrary, given neighborhood of x. In N_x there exists at least one point c of C. Let a positive number ϵ be given, and let N_c be a neighborhood of c such that N_c is contained in N_x. Since f is cliquish at c, there exists a neighborhood N contained in N_c (and hence contained in N_x) such that for every x_1 and x_2 of N it is true that

$$\rho[f(x_1), f(x_2)] < \epsilon.$$

Since N is contained in N_x, and since N_x was an arbitrary neighborhood of x, f is cliquish at x. But x was an arbitrary, given point of X. Therefore f is cliquish at every point of X. In other words f is cliquish.

As a direct consequence we have the result to which we alluded above that every pointwise non-neighborly function is cliquish.

THEOREM II. *The set of points at which a pointwise cliquish function is cliquish is nowhere dense.*

Proof. Suppose the set C where f is cliquish were not nowhere dense in the domain of definition X of f. Then there would exist at least one neighborhood N such that C would be everywhere dense in N. Then f would be cliquish on a dense set in N. Hence by the last theorem f would be cliquish at every point of N. This contradicts the hypothesis that the set of points where f is non-cliquish is everywhere dense in X.

It might be of interest to note that a type of converse of this theorem holds. That is, for every set S which is nowhere dense in an interval (a, b) there exist pointwise cliquish functions which are cliquish on S and non-cliquish at each

point of the everywhere dense complement of the closure of S in (a, b). An example of such functions can be constructed in the following way. The complement of the closure of S in (a, b) is an everywhere dense set which consists of the points of a denumerable number of non-overlapping open intervals. On each of these intervals the function is defined as follows: Let (α, β) be such an interval. Then

$$\begin{aligned} f(x) &= (x - \alpha) \quad \text{if} \quad \alpha < x \leqq \tfrac{1}{2}(\alpha + \beta), \quad \text{and } x \text{ is rational,} \\ &= -(x - \beta) \quad \text{if} \quad \tfrac{1}{2}(\alpha + \beta) < x < \beta, \quad \text{and } x \text{ is rational,} \\ &= 0 \qquad\quad \text{if} \quad \alpha < x < \beta \quad \text{and } x \text{ is irrational.} \end{aligned}$$

For each point x of the closure of S, $f(x) = 0$. This function is easily seen to be continuous, and hence cliquish, at each point of S. On every open interval (α, β) of the everywhere dense complement of S in (a, b), $f(x)$ is totally discontinuous and non-cliquish.

THEOREM III. *The limit $F(x)$ of a sequence of cliquish functions can be non-cliquish at every point of its domain of definition.*

Proof. Let $F(x) = 1$ if x is a rational number in the open interval $(0, 1)$, and let $F(x) = 0$ if x is an irrational number in $(0, 1)$. This function is non-cliquish at each point of $(0, 1)$. It is, however, the limit of the sequence $\{F_n(x)\}$, where

$$F_n(x) = \sum_{q=2}^{q=n} f_q(x),$$

and where the functions $f_q(x)$ are defined as follows:

$$f_q(x) = 1, \text{ if } x = p/q,\ p < q, \text{ and } p \text{ and } q \text{ are relatively prime}$$

positive integers,

$$f_q(x) = 0, \text{ if } \quad x \neq p/q.$$

Each $F_n(x)$ is obviously cliquish at each point of $(0, 1)$.

The three theorems stated illustrate how different the point property of cliquishness of a function is from the properties of continuity and neighborliness. The corresponding statements for continuous and neighborly functions are false. Thus for example, it is not true that every function, which is continuous on a dense set, is continuous, nor is every function, which is neighborly on a dense set, neighborly. As opposed to Theorem II we have for continuous functions the result that the set of points at which a pointwise continuous function is continuous may be everywhere dense. Instead of Theorem III there is the well known result that the limit of a sequence of continuous (neighborly) functions is at most pointwise discontinuous (non-neighborly).

The next two theorems and their proofs are analogues of similar theorems

and proofs of the theory of pointwise discontinuous and neighborly functions.

THEOREM IV. *The points of discontinuity of every cliquish function constitute a set of Baire's first category.*

THEOREM V. *Every cliquish function is at most pointwise discontinuous.*

COLLEGIATE MATHEMATICS AND THE NATIONAL SCIENCE FOUNDATION

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Having recently completed a short term as Acting Program Director for Mathematics in the National Science Foundation, I should like to make some remarks on the relation of collegiate mathematics to the Foundation. By collegiate mathematics I mean the professional activities associated with undergraduate and beginning graduate teaching, whether in a college or university.

The National Science Foundation Act of 1950 (Public Law 507—81st Congress) directed the Foundation in part "to develop and encourage the pursuit of a national policy for the promotion of basic research and education in the sciences; to initiate and support basic scientific research in the mathematical, physical, medical, biological, engineering and other sciences, by making contracts or other arrangements (including grants, loans and other forms of assistance) for the conduct of such basic scientific research and to appraise the impact of research upon industrial development and the general welfare; . . . ; to award scholarships and graduate fellowships in the mathematical, physical, medical, biological and other sciences; . . . " Note that this directive includes the strengthening of education in the sciences. However, we can assume that it does not mean that the Government will enter into the field of education as such.

It is well recognized that collegiate mathematics will be stronger if the teachers continue in active scholarship and research in their fields. If teaching duties interfere too much, the mathematician needs free time to carry out his research. This sort of research is supported by various government agencies, including the National Science Foundation, on a basis of proposals which describe individual research projects. If the proposal is favorably reviewed by mathematical consultants and there are funds, a grant is made to the institution to permit the work of the scientist to be carried out. In mathematics the principal item of the budget of any such proposal is the salary of the investigator for the time to be spent on the work. This is always determined by the investigator's regular academic salary. For example, if the academic salary covers service for nine months,

the full time summer research salary may be as much as one third the academic salary. The payment of a fraction of the salary of a man of professorial rank during the academic year for part-time research, when accompanied by a corresponding reduction in the total service load of the investigator, is still being considered by the National Science Foundation, although there are doubts as to the wisdom of it. These doubts do not apply to the relief of junior research personnel.

Before preparing a proposal for a research grant, the investigator should consult the "Guide for the Submission of Research Proposals" obtainable from the National Science Foundation, Mathematical, Physical and Engineering Sciences Division, Washington 25, D. C. The formal proposal should come from the administrative office of the institution. The budget should include an allowance for the publication which will arise from the research. It may include necessary travel in the United States and may include an item for the indirect costs of the institution not exceeding 15 per cent of the total. It is feasible for the investigator to write directly to the Program Director for Mathematics with a tentative proposal before submitting a formal one. The above describes the individual, short-term research proposal. A continuing group project having a similar effect to a departmental block grant is being described by the author in an article submitted to the Bulletin of the American Mathematical Society.

I have never seen a proposal asking for assistance in conducting a program of undergraduate honors work for gifted young people. Most colleges lack the funds to carry out such a program because the direction of these young people takes more time and is more costly than conventional teaching. But this is something which would obviously contribute much to the welfare of science in the future. Hence a department which has a tradition of some accomplishment in this activity might appropriately submit a proposal for an "undergraduate research" program. The needs might be some fraction of the time of the staff to be devoted to direction of the original investigations of the students selected to work for honors in mathematics. Also, in particular cases, some undergraduate research assistantships (don't call them scholarships) will be needed to permit the honors men to replace part time jobs by earning a comparable amount of money in their scientific work. It would be fine if there were an inexpensively published journal for publishing the better examples of these undergraduate researches, perhaps edited by graduate students.

There are other proper activities of mathematicians in a college or university which would contribute much towards the health of mathematical and scientific research. It will take some imagination to cast these good research-supporting activities in the form of proposals which ask for support in terms which government agencies can consider. One such area which should be explored, in my opinion, is the service of mathematicians as consultants to the scientific research groups in their own institutions. Under present circumstances the institution cannot ordinarily recognize this potentially valuable work by reduction of

normal duties and the mathematical profession does not recognize the work as a substitute for mathematical research. But when there is a mathematician who likes this sort of thing and he is good at it, he can be very valuable. Perhaps in applying for support for the general mathematical consultant, his work might be linked with the teaching of the intermediate level service courses in applied mathematics which are severely limited in a small institution.

Still another domain which needs cultivation in the ranks of mathematicians as a whole is what might be termed developmental mathematical research. It is rather generally recognized that there is an unhealthy tendency, brought about in part by the limitations of publication, to give the exalted title of RESEARCH only to the construction of the very general and abstract theories. The working out of particular problems within these theories is usually relegated to "masters' theses," even though there is no reasonable chance that so immature a student could comprehend what he would need to know. It would be good if there were more mathematicians who would unabashedly commit themselves to this sort of developmental research. It would bring more immature mathematicians into research which would be appropriate to their interests and would contribute to the understanding and use of the general theories laid out by the few real innovators. It would tend to preserve problem-solving ability in American mathematics and in some cases could be expected to lead into new general theory. A cheap journal, associated with an abstracting and evaluation service, perhaps a microcard or microfilm journal, would be a help in recording the results without undue bulging of the libraries or taxing the finances of publications. Even as things are now, there is nothing to exclude this type of proposal.

Some things which are probably not supportable by government agencies are: regional scholarships, sabbaticals, teaching, preparation of new courses, writing of textbooks (though the research which goes into a book may be supported), travel abroad (except under Fulbright Act or travel to a scientific meeting abroad), block grants to an institution without specific determination of the purpose and use and accountability for the results of the work, and anything continuing in perpetuo. Research which derives its scientific interest from a particular geographic location, however meritorious, will usually meet with difficulty. In general, the words "education" and "scholarships" or "fellowships" should be avoided in research proposals, because these are things which are nationalized if supported at all, and come under the Scientific Personnel and Education Division of the Foundation.

Finally, I should like to make a plea to all the members of this Association to submit proposals to the Foundation expressing your real needs in research or support of research. Do not be disturbed if these proposals must be rejected for lack of funds or other reasons, but consider that your time spent was a public service and a service to mathematics. For only from proposals can the real needs of mathematics and mathematicians be determined.

A SUPPLEMENTARY NOTE TO A 1946 ARTICLE ON FERMAT'S LAST THEOREM

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Concerning the relation

$$x^l + y^l + z^l = 0, \tag{1}$$

where l is an odd prime >2, the writer (this MONTHLY, vol. 53, pp. 555–578, which article we shall refer to here as (I)) gave a number of references to papers which included results concerning (1) which enabled us to state that Fermat's well known statement is true for certain classes of exponents l in (1) and none of these classes was the null class. In error I omitted several papers which belong to the type I definitely meant to treat. I shall now give reports on these latter papers.

H. J. S. Smith [1] gave a report of some of the principal results which had been found concerning (1) prior to the year 1860, particularly those of Kummer.

K. Hensel [2] commented on Kummer's various results concerning Fermat's last theorem.

Got [3] reproduced with explanations Kummer's [4] memoir of 1857 on Fermat's last theorem.

J. McDonnel [5] proved that if p is an odd prime and

$$x^p + y^p + z^p = 0 \tag{2}$$

with $p>2$ and if

$$(yz + zx + xy, p) = 1, \qquad (x, y, z, p) = 1,$$

and

$$\frac{r^{p-1} - 1}{p} = q(r),$$

$$x^2 - yz \equiv 0 \pmod{r},$$

then it follows that

$$q(r) \equiv 0 \pmod{p}.$$

Also, if (2) holds with

$$(x(y - z)(x^2 + yz), p) = b, \qquad (x, y, z, p) = 1$$

and

$$x^2 + yz \equiv 0 \pmod{r_1},$$

then

$$q(r_1) \equiv 0 \pmod{p}.$$

L. Holzer [6] showed that if (1) holds with l an odd prime, and

$$(x, y, z, l) = 1, \qquad a = xy - z^3 \not\equiv 0 \pmod{l}, \qquad a \equiv 0 \pmod{r},$$

then

$$q(r) \equiv 0 \pmod{l}.$$

Also, if

$$((x - y)(xy + z^2), l) = 1, \qquad xy + z^2 \equiv 0 \pmod{r_1},$$

then

$$q(r_1) \equiv 0 \pmod{l}.$$

Compare this with our preceding reference.

H. Hasse [7] gave an account of some of the principal results for the first case of Fermat's last theorem, including some of those obtained by Kummer, Mirimanoff, and Furtwängler. In particular, he obtained the Kummer criteria, that is, if (1) is satisfied with l an odd prime, and $xyz \not\equiv 0 \pmod{l}$, then

$$B_n f_{l-2n}(t) \equiv 0 \pmod{l}; \qquad n = 1, 2, \cdots, (l - 3)/2;$$

where if $m > 1$,

$$f_m(t) = \sum_{i=1}^{l-1} i^{m-1} t^i; \; t = -x/y, -y/x, -y/z, -z/y, -x/z, -z/x,$$

and B_n is the nth Bernoulli number, $B_1 = 1/6$, $B_2 = 1/30$, *etc.* The method he used to obtain this is different than that which was employed by Kummer, Hasse using explicit expressions for the power characters of units in a cyclotomic field.

M. Krasner [8] using the criteria of Kummer in the form originally given by the latter instead of Mirimanoff's form as given in our last reference, proved the following result:

There exists a number l_0, such that for $l \geqq l_0$, the existence of three integers, x, y, z, prime to l and such that

$$x^l + y^l + z^l = 0,$$

implies that l is a divisor of $[\sqrt{\log l}]$ consecutive Bernoulli numbers, the last of which is $B_{(l-3)/2}$. Here the symbol $[k]$ means the greatest integer in k.

In (I) page 575, paragraph 5, the writer discussed various results which led him to the opinion that Fermat's last theorem is true in Case I, that is, where $xyz \not\equiv 0 \pmod{l}$ in (1). In March of last year I received a letter from Dr. Krasner, which refers to his result quoted above, where l is replaced by p. I quote herewith his letter in part:

"Concerning your discussion of the truth of Fermat's Theorem, I think as

you that in the Case I it is certainly true. Your argument is certainly in order. (This statement refers to the reference given above to the present writer.) Even the numbers p not satisfying my criterion given in Theorem VIII of your article must be very exceptional. But I think that my preceding result given in the *Comptes Rendus* of 1934 furnishes even much stronger arguments in this sense. It seems quite unlikely that all the $[\sqrt[3]{\log p}]$ consecutive Bernoullian numbers

$$B_{(p-1)/2-i}; \qquad 0 < i \leqq \sqrt[3]{\log p}$$

are divisible by p, if only p is not too small. If you admit that the probability for an unknown Bernoullian number to be divisible by p is $1/p$, such a divisibility has only the probability

$$\left(\frac{1}{p}\right)^{[\sqrt[3]{\log p}]} \approx \frac{1}{p^{\sqrt[3]{\log p}}},$$

and a simple calculation shows that the mathematical expectation of the number of primes $p \geqq n$ satisfying this condition does not exceed

$$\frac{2}{n^{\sqrt[3]{\log n}}}.$$

When we use a little different and very likely hypothesis that, if f_p is the frequency of the Bernoullian numbers divisible by p,

$$\frac{\sum_{p \leq x} p f_p}{\pi(x)} \to 1,$$

the result is not very different, with only, maybe, a greater dispersion."

H. S. Vandiver [9] gave without proof the result that if (1) is satisfied in Case I, then, if $l_1 = (l-3)/2$,

$$B_s \equiv 0 \pmod{l^2}, \qquad (i = 1, 2, 3, 4, 5, 6; s = n_i(l_1 + 1) - i),$$

where the n's each range independently over all positive integral values. He indicated later [10] a proof of this statement as well as discussing a method of examining these criteria in special cases.

References

1. Smith, Report of the British Association, Part II, 1860, pp. 147–153, Collected Mathematical Works, Oxford Press; Collected Math. Papers, vol. 1, 1894, pp. 131–137.

2. Hensel, Abhandlungen zur Geschichte der Mathematik, vol. 29, 1910, pp. 18–32.

3. Got, Annales de la faculté des sciences de Toulouse, vol. 25, 1911, pp. 21–62.

4. Kummer, Abhandlungen der K. Akademie der Wiss. zu Berlin, Math.-Phys. Klasse, 1857, pp. 41–74. Abstract in Berlin Monatsberichte, 1857, pp. 275–282.

5. McDonnel, Bull. Amer. Math. Soc., vol. 36, 1930, pp. 553–558.

6. Holzer, Jour. für Math., vol. 173, 1935, pp. 118–119.

7. Hasse, Bericht über neuere Untersuchungen, etc., Jahresber. Deuts. Math. Verein., II, 1930, pp. 113–122.

8. Krasner, On the first case of Fermat's last theorem, Comptes Rendus, 199, 1934, p. 256. As stated in the first paragraph and second footnote of (I), we were confining ourselves in that paper to citing papers giving results showing that (1) was impossible for certain classes of prime exponents, and it was possible to show that none of these classes was the null-class. No references were listed in the bibliography unless the article cited included results of the type just mentioned, with the exception of M. Krasner's article here reviewed. However, mistakenly, the writer put Krasner's paper in the wrong category.

9. Vandiver, Proc. Nat. Acad. Sci., vol. 16, 1930, p. 298; Bull. Amer. Math. Soc., Feb., 1934, p. 124.

10. Vandiver, Proc. Nat. Acad. Sci., vol. 27, 1941, p. 82; Duke Math. Journal, vol. 5, 1939, pp. 425–427.

ON CEVIANS OF A TRIANGLE*

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1. Introduction. The lines, one from each vertex of a triangle, which cut the opposite sides of that triangle in the ratio of the nth powers of the adjacent sides, possess interesting properties. It seems that only the simple cases corresponding to the medians, bisectors, symmedians, that is to $n=0$, 1, 2, have been considered. This note takes up the general case and then arrives at a system of hyperbolic lines associated with a tetrahedron, for which J. Neuberg gave two particular cases in his *Mémoire sur le tétraèdre* (1884).

2. Notation. Let a, b, c $(a+b+c=2p)$, designate the sides BC, CA, AB of a triangle; (O), (O_9), (I), (I_i), $(i=a, b, c)$, denote the circumcircle, the nine-point circle, the incircle and the excircles of respective radii R, $R/2$, r, r_i; AA', BB', CC', G, H, K, indicate the altitudes, centroid, orthocenter and symmedian point of the triangle $T\equiv ABC$ oriented in the sense ABC.

3. Metric relations. From the theorem of Ceva it follows that the lines AA_1, BB_1, CC_1 which cut the sides BC, CA, AB in the ratios

$$(1) \qquad BA_1/A_1C = c^n/b^n, \qquad CB_1/B_1A = a^n/c^n, \qquad AC_1/C_1B = b^n/a^n$$

are concurrent at a point P_n. The harmonic conjugates A_1', B_1', C_1' of A_1, B_1, C_1 with respect to B and C, C and A, A and B, are collinear as they are on the trilinear polar of P_n. Likewise the sets of points (B_1, C_1, A_1'), (C_1, A_1, B_1'), (A_1, B_1, C_1') are on the trilinear polars of the associates of P_n with respect to the triangle T. The associates of a point of normal (or barycentric) coordinates (x, y, z) are the points of coordinates $(-x, y, z)$, $(x, -y, z)$, $(x, y, -z)$.

Furthermore in magnitude and sign

* Translated from the French by W. E. Byrne.

$$(2)\qquad \begin{aligned} BA_1 &= ac^n/(b^n + c^n), & A_1'B &= ac^n/(b^n - c^n),\\ A_1C &= ab^n/(b^n + c^n), & A_1'C &= ab^n/(b^n - c^n). \end{aligned}$$

Also

$$(3)\qquad AP_n/P_nA_1 = AC_1\cdot BC/C_1B\cdot AC_1 = (b^n + c^n)/a^n$$

so that

$$(4)\qquad \begin{aligned} AP_n &= bc[(b^{n-2} + c^{n-2})(b^n + c^n) - a^2(bc)^{n-2}]^{1/2}/(a^n + b^n + c^n)\\ &= a^n\cdot P_nA_1/(b^n + c^n). \end{aligned}$$

Similar formulae may be obtained by cyclical permutation. Thus the barycentric coordinates of P_n, and its associates P_a, P_b, P_c are (a^n, b^n, c^n), $(-a^n, b^n, c^n)$, $(a^n, -b^n, c^n)$, $(a^n, b^n, -c^n)$ with respect to T as the reference triangle.

An application of Stewart's theorem to T and to AA_1, AA_1', BB_1, BB_1', CC_1, CC_1' gives

$$(5)\qquad \overline{AA_1}^2 = (bc)^2[(b^{n-2} + c^{n-2})(b^n + c^n) - a^2(bc)^{n-2}]/(b^n + c^n)^2$$

$$(6)\qquad \overline{AA_1'}^2 = (bc)^2[(b^{n-2} - c^{n-2})(b^n - c^n) + a^2(bc)^{n-2}]/(b^n - c^n)^2$$

as well as like formulas obtained by cyclical permutation.

THEOREM. *The square of the distance of any point Q from P_n is given by*

$$(7)\qquad \begin{aligned} \overline{QP_n}^2 &= \sum a^n\overline{QA}^2/(\sum a^n) - (abc/\sum a^n)^2\cdot\sum (bc)^{n-2}\\ &= \sum a^n\overline{QA}^2/(\sum a^n) - D_n \end{aligned}$$

where D_n is a constant independent of the position of Q [1].

Proof. (7) follows from (5) and an application of Stewart's theorem to triangles BQC, AQA_1.

Likewise

$$(8)\qquad \begin{aligned} \overline{QP_a}^2 &= (-a^n\overline{QA}^2 + b^n\overline{QB}^2 + c^n\overline{QC}^2)/(-a^n + b^n + c^n)\\ &\quad + (abc)^2/(-a^n + b^n + c^n)^2[-(bc)^{n-2} + (ca)^{n-2} + (ba)^{n-2}]\\ &= (-a^n\overline{QA}^2 + b^n\overline{QB}^2 + c^n\overline{QC}^2)/(-a^n + b^n + c^n) + D_a, \end{aligned}$$

where D_i are constants independent of the position of the point Q. For $n=0, 1, 2$ these reduce to

$$\begin{aligned} D_0 &= (a^2 + b^2 + c^2)/9, & D_a &= -a^2 + b^2 + c^2, \cdots\\ D_1 &= 2Rr, & D_a &= 2Rr_a, \cdots\\ D_2 &= 3(abc)^2/(a^2 + b^2 + c^2)^2, & D_a &= (abc)^2/(-a^2 + b^2 + c^2)^2, \cdots. \end{aligned}$$

Thus the measures of the medians, bisectors, symmedians and the squares of the distances of O and H from G, I, K can be obtained. For $n=1$ and $Q=O$,

formulas (7), (8) reduce to the Feuerbach relations since (7) gives

$$\overline{O_9I}^2 = \sum a\,\overline{O_9A}^2/2p - D_1.$$

But

$$\overline{O_9A}^2 = (R^2 + b^2 + c^2 - a^2)/4 = (R^2 + 2bc\cos A)/4$$
$$\overline{O_9B}^2 = (R^2 + 2ca\cos B)/4$$
$$\overline{O_9C}^2 = (R^2 + 2ab\cos C)/4$$

so that

$$\sum a\cdot\overline{D_9A}^2/4p = (2pR^2 + 2abc\sum\cos A)/4p$$
$$= (R^2 + 4rR + 4r^2)/4$$

and

$$\overline{O_9I}^2 = (R^2 + 4Rr + 4r^2)/4 - 2Rr = (R - 2r)^2/4.$$

Likewise

$$\overline{O_9I_a}^2 = (R + 2ra)^2/4.$$

Furthermore it follows from (2) that the generalized circles of Apollonius (O_i) described on A_1A_1', B_1B_1', C_1C_1' as diameters are orthogonal to (O).

Since

$$A_1A_1' = 2abc(bc)^{n-1}/(b^{2n} - c^{2n})$$

then

$$A_1A_1'/a^{n-1} + BB_1'/b^{n-1} + C_1C_1'/c^{n-1} = 0.$$

The measures of the diameters of the circles (O_i) are independent of the lengths of the sides of the triangle on which the centers O_i are situated, and the same holds for the powers

$$(b^{2n+2} - c^{2n+2})/(b^{2n} - c^{2n}), \cdots$$

of the vertices A, B, C for the circles symmetric to the (O_i) with respect to the midpoints of BC, CA, AB. If $n=1$, these powers are equal to b^2+c^2, c^2+a^2, a^2+b^2.

4. Associated circles and triangles. THEOREM. *The circle $A_1B_1C_1$ intercepts segments x, y, z on BC, CA, AB which satisfy the equation*

$$\frac{x}{a^{n-1}} + \frac{y}{b^{n-1}} + \frac{z}{c^{n-1}} = 0. \tag{9}$$

Proof. If we designate by A_3, B_3, C_3 the second points of intersection of the circle $A_1B_1C_1$ with BC, CA, AB and if $a \geqq b \geqq c$, then

$$AB_1 \cdot AB_3 = AC_1 \cdot AC_3$$

and like relations for the other vertices of T. These relations with (2) give

$$bc^n[bc^n + (c^n + a^n)y]/(c^n + a^n)^2 = cb^n[cb^n - (a^n + b^n)z]/(a^n + b^n)^2$$
$$ca^n[ca^n + (a^n + b^n)z]/(a^n + b^n)^2 = ac^n[ac^n - (b^n + c^n)x]/(b^n + c^n)^2$$
$$ab^n[ab^n + (b^n + c^n)x]/(b^n + c^n)^2 = ba^n[ba^n - (c^n + a^n)y]/(c^n + a^n)^2.$$

If the above equations are multiplied by a^{2n}, b^{2n}, c^{2n} and added, (9) follows.

If $n=1$, 2 the cevians are the bisectors and the symmedians of T, so

$$x + y + z = 0 \qquad (n = 1) \qquad [2]$$
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 \qquad (n = 2).$$

Relations like (9) hold for the segments intercepted by the circles $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ on BC, CA, AB.

The lines AA_3, BB_3, CC_3 are concurrent in a point P_n' [3]. The points A_1, B_1, C_1, A_1', B_1', C_1' are the vertices of a complete quadrilateral (Q) which has T as its diagonal triangle. The Miquel point M of (Q), which is common to the circles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$, is the focus of the parabola inscribed in (Q). Furthermore the centers of the equilateral hyperbolas circumscribed about the quadrangles $ABCP_n$, and $ABCP_n'$ are distinct points on (O_9) as they are the orthopoles f and f', with respect to T, of the diameters OP_n and OP_n' of the circle ABC. f' coincides with M. The above may be summarized as follows:

THEOREM. *The orthocenters of the triangles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ are on the orthopolar Δ with respect to T of the Miquel point M of (Q). Δ is a diameter of the circle ABC. The Simson line of the point M, with respect to the complementary triangle t of T, is common to the triangles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ and the orthopoles (with respect to t) of the sides of these triangles are on the line Δ.*

When $n=1$, P_1 coincides with the incenter of T and P_1' is on the Kiepert hyperbola circumscribed about T. Hence the circles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$ meet at the center of the Kiepert hyperbola. Each of these circles contains one of the Feuerbach points of T [4]. The Simson line with respect to t of the Miquel point M of quadrilateral (Q) is common to the orthic triangle $A'B'C'$ and the triangles $A_1B_1C_1$, $A_1B_1'C_1'$, $B_1C_1'A_1'$, $C_1A_1'B_1'$. Also the orthopoles, with respect to t, of the sides of these triangles are on the Brocard diameter OK of T. See reference [5] for some special cases.

5. Hyperbolic systems. Consider an arbitrary tetrahedron $T \equiv ABCD$ with edges BC, DA, CA, DB, AB, DB of lengths a, a', b, b', c, c'. Let (a_1, b_1, c_1), (a_2, c_1', b_1'), (c_2', a_1', b_2), (a_2', c_2, b_2') designate the feet of the cevians which

divide the sides of the oriented triangles ABC, BCD, CDA, DAB in the ratios

$$Ba_1/a_1C = c^n/b^n, \qquad Cb_1/b_1A = a^n/c^n, \qquad Ac_1/c_1B = b^n/a^n,$$

$$Ba_2/a_2C = b'^n/c'^n, \qquad Cb_2/b_2A = c'^n/a'^n, \qquad Ac_2/c_2B = a'^n/b'^n, \tag{10}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

Let (A_n, B_n, C_n, D_n), (A_n', B_n', C_n', D_n') denote the points of intersection of these sets of cevians in the triangles ABC, BCD, CDA, DAB.

THEOREM. *The squares of the segments AA_n, BB_n, CC_n, DD_n and AA_n', BB_n', CC_n', DD_n' are given by*

$$\overline{DD_n}^2 = [\textstyle\sum a^n \cdot \sum (aa')^n - (abc)^2 \sum (bc)^{n-2}]/(\sum a^n) \tag{11}$$

$$\overline{DD_n'}^2 = \left[\sum \frac{1}{a^n} \cdot \sum \frac{1}{a'^{n-1}} - \sum \frac{a^2}{(bc)^n}\right] \Big/ \left(\sum \frac{1}{a^n}\right). \tag{12}$$

Formulas (11), (12) may be obtained by successive applications of Stewart's theorem.

THEOREM. *AA_n, BB_n, CC_n, DD_n are rulings of one system and AA_n', BB_n', CC_n', DD_n' are rulings of a second system of an hyperboloid.*

Proof. Lines AB_n, AC_n, AD_n meet the edges DC, DB, BC at points c_2', b_2', a_2 such that

$$Bb_2'/b_2'D = c^n/a'^n, \qquad Dc_2'/c_2'A = a'^n/b^n, \qquad Ca_2/a_2B = b^n/c^n \tag{13}$$

and Bc_2', Cb_2', Da_2 meet at A_n'. Analogous statements hold for the points B_n', C_n', D_n' of the other faces. Hence the line AA_n', for instance, meets AA_n, BB_n, CC_n, DD_n, which are four rulings of one system of an hyperboloid; AA_n', BB_n', CC_n', DD_n' are rulings of the other system.

COROLLARY. *If AA_n, BB_n, CC_n, DD_n coincide with AA_n', BB_n', CC_n', DD_n', then they are concurrent and the tetrahedron T is isodynamic, and conversely.*

Proof. The necessary and sufficient condition that b_1' and b_2' coincide as well as the pairs a_1, a_2 and c_1', c_2' is that

$$(aa')^n = (bb')^n = (cc')^n$$

and conversely. But these equations follow from (10) and (13). So in this case $AA_n = AA_n', \cdots$ and they are concurrent, and T is isodynamic.

COROLLARY. *If T is isodynamic the lines a_1a_1', b_1b_1', c_1c_1' meet at the point of intersection L of AA_n, BB_n, CC_n, DD_n.*

Proof. Line a_1a_1' situated in the planes $Ba_1'C$ and Aa_1D meets the lines AA_n, DD_n and BB_n, CC_n in these planes and hence goes through L.

COROLLARY. *The trilinear polars of the points A_n, B_n, C_n, E_n with respect to the faces BCD, CDA, DAB, ABC of an isodynamic tetrahedron are situated in the polar plane with respect to T of the point L.*

In like manner the following more general theorem may be obtained [6].

THEOREM. *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_1', \lambda_2', \lambda_3'$ be associated with the edges BC, CA, AB, DA, DB, DC of a tetrahedron. Let A_0, B_0, C_0, D_0 be the points of the planes of the faces BCD, CDA, DAB, ABC for which the barycentric coordinates are proportional to the numbers λ associated with the sides of the corresponding face. AA_0, BB_0, CC_0, DD_0 are four rulings of one system of an hyperboloid. They are concurrent if and only if*

$$\lambda_1\lambda_1' = \lambda_2\lambda_2' = \lambda_3\lambda_3'.$$

Particular cases. $n=1$. The points A_n, B_n, C_n, D_n are the incenters of the faces *BCD, CDA, DAB, ABC* of *T*. Thus the lines joining the vertices of an arbitrary tetrahedron to the incenters of the opposite faces are four rulings of an hyperboloid [7]. This may be generalized as follows:

THEOREM. *If perpendiculars are erected to the faces of tetrahedron T at the centers A_1, B_1, C_1, D_1 of faces BCD, CDA, DAB, ABC and segments A_1A_2, B_1B_2, C_1C_2, D_1D_2 are laid off toward the exterior (interior) of T proportional to the radii of the inscribed circles, the lines AA_2, BB_2, CC_2, DD_2 are rulings of an hyperboloid.*

Proof. Let us designate by $AA' = h_a$ the altitude of *T* drawn from *A* and put $A_1A_2 = mr_a$, r_a being the radius of the inscribed circle of face *BCD*, *m* arbitrary. The distances of A' from the edges *BC*, *CD*, *DB* are $h_a \cot \alpha$, $h_a \cot \gamma'$, $h_a \cot \beta'$ where α is the measure of the dihedral angle of edge *a*. AA_2 meets $A'A_1$ at a point A_3 which divides the segment $A'A_1$ in the ratio

$$A'A_3/A'A_1 = h_a/(h_a + mr_a) = k.$$

The distances of A_3 from the edges *BC*, *CD*, *DB* are

$$kr_a(m \cot \alpha + 1), \qquad kr_a(m \cot \gamma' + 1), \qquad kr_a(m \cot \beta' + 1).$$

A_3 has

$$\lambda_1 = a(m \cot \alpha + 1), \qquad \lambda_3' = c'(m \cot \gamma' + 1), \qquad \lambda_1' = b'(m \cot \beta' + 1)$$

as barycentric coordinates with respect to triangle *BCD*. In like manner the barycentric coordinates of B_3, C_3, D_3, analogous to A_3, with respect to faces *CDA*, *DAB*, *ABC* are

$$(\lambda_2', \lambda_1', \lambda_2), \qquad (\lambda_1', \lambda_3, \lambda_2'), \qquad (\lambda_1, \lambda_2, \lambda_3).$$

The proof is then reduced to the previous theorem.

It may be noted that since

$$m = A_1A_2/r_a = B_1B_2/r_b = C_1C_2/r_c = D_1D_2/r_d,$$

the planes (A_2BC, A_2CD, A_2DB), (B_2CD, B_2DA, B_2AC), (C_2DA, C_2AB, C_2BD), (D_2AB, D_2BC, D_2CA) are equally inclined with respect to the faces BCD, CDA, DAB, ABC, respectively. The configuration of lines AA_2, BB_2, CC_2, DD_2 is then like that of the concurrent lines MM_1, NN_1, PP_1 joining the vertices of a triangle to the vertices of the directly similar isosceles triangles MP_1N, NM_1P, PN_1M. We find thus a particular case of the following theorem: [8], [9].

THEOREM. *Given a tetrahedron $ABCD$ and a surface (S) of the second class, consider a face ABC for instance, draw through each edge a plane which has a given cross-ratio k with the face and the two tangent planes of (S) through that edge. The three planes thus defined meet at a point D'. Likewise points A', B', C' are determined. The lines AA', BB', CC', DD' are hyperbolic. If $k=-1$, D' coincides with the pole of the plane ABC with respect to (S).*

$n=2$. In this case A_n, B_n, C_n, D_n are the symmedian points (Lemoine points) of the faces BCD, CDA, DAB, ABC. The lines joining the vertices of a tetrahedron to the symmedian points of the opposite faces are hyperbolic [10].

References

1. C. Thiry, Bull. de l'Académie Royale, 1891, p. 471.
2. B. E. Lawrence, Mathematical Gazette, 1945, p. 18.
3. E. Weber, Nouvelles Annales, 1906, p. 343.
4. E. Weber, L'Enseignement mathématique (Geneva), 1907, p. 219. R. Guillotin, Mathesis, t. 59, 1950, p. 93.
5. Deprez, Mathesis, 1905, p. 156.
6. V. Thébault, Mathesis, 1931, p. 51.
7. J. Neuberg, Mémoire sur le tétraèdre, p. 23.
8. K. Dochlemann, Archiv für Math. u. Physik, (2), 17, 1900, pp. 157–174.
9. L. Klug, ibid (3), 8, 1905, pp. 157–161.
10. J. Neuberg, ibid.

MATHEMATICAL NOTES

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NOTES ON MATRIX THEORY—II

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The purpose of this note is to indicate how a well-known formula of integral calculus may be used to derive some results of interest pertaining to the determinant of a positive definite matrix A.

If $A=(a_{ij})$, $i, j=1, 2, \cdots, n$, is a positive definite matrix we have

$$\frac{C_n}{|A|^{n/2}} = \int_{-\infty}^{\infty} e^{-\sum_{i,j=1}^{n} a_{ij}x_ix_j} \prod_{i=1}^{n} dx_i, \tag{1}$$

where $|A|=|a_{ij}|$ and $C_n=(\sqrt{\pi})^n$. The proof of (1) follows readily if we transform the quadratic form into a sum of squares, $\sum_{k=1}^{n}\lambda_k x_k^2$ by means of an orthogonal transformation, where λ_k are the characteristic roots of A, and use the fact that $|A|=\prod_{k=1}^{n}\lambda_k$.

As a first application, let us prove the known result,

$$|\alpha A + (1-\alpha)B| \geqq |A|^{\alpha}|B|^{1-\alpha}, \tag{2}$$

if A and B are positive definite and $0 \leqq \alpha \leqq 1$. We have

$$\frac{C_n}{|\alpha A + (1-\alpha)B|^{n/2}} = \int_{-\infty}^{\infty} e^{-\alpha\sum_{i,j=1}^{n} a_{ij}x_ix_j-(1-\alpha)\sum_{i,j=1}^{n} b_{ij}x_ix_j} \prod dx_i. \tag{3}$$

Applying Hölder's integral inequality with $0<\alpha<1$, $p=1/\alpha$, $p'=1/(1-\alpha)$, we obtain

$$\begin{aligned}\frac{C_n}{|\alpha A + (1-\alpha)B|^{n/2}} &\leqq \left(\int_{-\infty}^{\infty} e^{-\sum a_{ij}x_ix_j}\prod dx_i\right)^{\alpha}\left(\int_{-\infty}^{\infty} e^{-\sum b_{ij}x_ix_j}\prod dx_i\right)^{1-\alpha} \\ &\leqq \left(\frac{C_n}{|A|^{n/2}}\right)^{\alpha}\left(\frac{C_n}{|B|^{n/2}}\right)^{1-\alpha},\end{aligned} \tag{4}$$

which is equivalent to (2).

As a second application, let us establish the classical result,

$$\prod_{i=1}^{n} a_{ii} \geqq |A|. \tag{5}$$

To illustrate the method, it is sufficient to consider the first non-trivial case, $n=3$. In (1) set successively $x_1=-x_1'$, $x_2=-x_2'$, $x_3=x_3'$ and add, obtaining

$$\begin{aligned}\frac{C_3}{|A|^{3/2}} = \int_{-\infty}^{\infty} & e^{-a_{11}x_1^2-a_{22}x_2^2-a_{33}x_3^2} \\ & \cdot\left[\frac{y_1y_2y_3 + y_1y_2^{-1}y_3^{-1} + y_1^{-1}y_2y_3^{-1} + y_1^{-1}y_2^{-1}y_3}{4}\right]\prod dx_i,\end{aligned} \tag{6}$$

where $y_1=e^{-a_{23}x_2x_3}$, $y_2=e^{-a_{13}x_1x_3}$, $y_3=e^{-a_{12}x_1x_2}$. The arithmetic-geometric mean inequality yields

$$\frac{y_1y_2y_3 + y_1y_2^{-1}y_3^{-1} + y_1^{-1}y_2y_3^{-1} + y_1^{-1}y_2^{-1}y_3}{4} \geqq 1. \tag{7}$$

Therefore

$$(8) \qquad \frac{C_3}{|A|^{3/2}} \geqq \int_{-\infty}^{\infty} e^{-a_{11}x_1^2 - a_{22}x_2^2 - a_{33}x_3^2} \prod dx_i = \frac{C_3}{(a_{11}a_{22}a_{33})^{3/2}},$$

which is equivalent to (5) for $n=3$.

A METHOD FOR FINDING PRIMES

JOHN THOMPSON, Yale University

The simplest way to show that there exists an infinite number of primes is probably by Euclid's method of taking the n first primes

$$(1) \qquad p_1 = 2, \cdots, p_n$$

and observe that the number

$$N = 1 + p_1 p_2 \cdots p_n$$

must be divisible by primes different from those in (1).

A slight variation of this method goes further and actually yields a method to determine new primes more explicitly from those given in (1). This depends on the observation that if one splits the primes in (1) up into two different groups

$$q_1, \cdots, q_a; \qquad r_1, \cdots, r_b$$

then the difference between the two products

$$(2) \qquad D = (q_1 \cdots q_a) - (r_1 \cdots r_b)$$

is also a number which cannot be divisible by any of the primes (1). Furthermore if the difference (2) should turn out to be less than $(p_n+2)^2$ the difference (2) is evidently a prime.

Following are some simple examples:

$$\begin{aligned} (11\cdot 5) - (2\cdot 3\cdot 7) &= 13 \\ (7\cdot 11) - (2\cdot 3\cdot 5) &= 47 \\ (2\cdot 3\cdot 11) - (5\cdot 7) &= 31 \\ (3\cdot 5\cdot 7) - (2\cdot 11) &= 83 \\ (3\cdot 13\cdot 17\cdot 23) - (2\cdot 5\cdot 7\cdot 11\cdot 19) &= 619 \\ (3\cdot 5\cdot 7\cdot 11\cdot 13) - (2\cdot 17\cdot 19\cdot 23) &= 157 \end{aligned}$$

One may ask whether a suitable choice of the products in (2) will always produce a prime. A wider range of possibilities is opened up by using exponents for the primes in (2).

AN INEQUALITY FOR REARRANGEMENTS

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Let $f_1(x), f_2(x), \cdots$ denote positive measurable functions on (0, 1) and $f_1^*(x), f_2^*(x), \cdots$ their equimeasurable decreasing rearrangements (see [1], [3]). For the work dealing with rearrangements, the following simple inequality is basic:

$$\int_0^1 f_1(x)f_2(x)dx \leqq \int_0^1 f_1^*(x)f_2^*(x)dx. \tag{1}$$

There are, however, also other combinations of $f_1, f_2, \cdots$ for which relations similar to (1) hold. One of these was given by Ruderman [2, Theorem II]. In this note we propose to determine, quite generally, necessary and sufficient conditions on a continuous function $\Phi(x, u_1, \cdots, u_n)$ defined for $0<x<1$, $u_k \geqq 0$, $k=1, 2, \cdots, n$, under which

$$\int_0^1 \Phi(x, f_1(x), \cdots, f_n(x))dx \leqq \int_0^1 \Phi(x, f_1^*(x), \cdots, f_n^*(x))dx \tag{2}$$

is satisfied for each set $f_k(x)$, $k=1, \cdots, n$, of positive bounded measurable functions on (0, 1). (We assume the $f_k(x)$ bounded in order to insure the existence of both integrals in (2).)

In inequalities containing values of the function Φ at different points, we shall omit those of the arguments $x, u_1, \cdots, u_n$ which take the same but arbitrary values. For a set I of indices i, $1 \leqq i \leqq n$, we put $U_I = \{u_i\}_{i\in I}$. We also put $U_I + U_I' = \{u_i + u_i'\}_{i\in I}$, if $U_I' = \{u_i'\}$.

THEOREM. *In order that Φ satisfy (2) it is necessary and sufficient that Φ have the properties*

$$\Phi(u_i + h, u_j + h) - \Phi(u_i + h, u_j) - \Phi(u_i, u_j + h) + \Phi(u_i, u_j) \geqq 0, \tag{3}$$

$$\int_0^\delta \{\Phi(x - t, u_i + h) + \Phi(x + t, u_i) - \Phi(x + t, u_i + h) - \Phi(x - t, u_i)\}dt \geqq 0 \tag{4}$$

for all $0<x<1$, $u_k \geqq 0$, $k=1, \cdots, n$, $h>0$, $0<\delta<x$, $\delta<1-x$, and $i\neq j$. If Φ has continuous second partial derivatives with respect to all variables, conditions (3), (4) are equivalent to

$$\frac{\partial^2\Phi}{\partial u_i \partial u_j} \geqq 0, \tag{3a}$$

$$\frac{\partial^2\Phi}{\partial x \partial u_i} \leqq 0. \tag{4a}$$

Proof. Suppose $0<a<1$, $0<\delta<a$, $\delta<1-a$, $i\neq j$. Define $f_i(x) = u_i + h_i$ for

$x \leqq a-\delta$ and $a<x \leqq a+\delta$ and $f_i(x)=u_i$ for other x, $f_j(x)=u_j+h_j$ for $x \leqq a$, $f_j(x)=u_j$ for $x>a$, further $f_k(x)=u_k$, $0<x<1$ for k different from i and j. Then the inequality (2) reduces to

$$\int_0^\delta \{\Phi(a-t, u_i+h_i, u_j+h_j) - \Phi(a+t, u_i+h_i, u_j) - \Phi(a-t, u_j, u_j+h_j) + \Phi(a+t, u_i, u_j)\}\,dt \geqq 0.$$

Putting here $h_j=0$, we obtain (4). Dividing through by δ and making $\delta \to 0$, we obtain (3).

To prove that the conditions are sufficient, we first deduce from (3) that for any two disjoint groups of indices I, J and h_i, $h_j \geqq 0$,

$$\Phi(U_I+H_I, U_J+H_J) - \Phi(U_I+H_I, U_J) - \Phi(U_I, U_J+H_J) + \Phi(U_I, U_J) \geqq 0. \tag{5}$$

From (3) we have

$$\Phi(u_i+sh, u_j+h) - \Phi(u_i+sh, u_j) - \Phi(u_i+(s-1)h, u_j+h) + \Phi(u_i+(s-1)h, u_j) \geqq 0.$$

Adding these relations for $s=1, 2, \cdots, p$ we deduce

$$\Phi(u_i+ph, u_j+h) - \Phi(u_i+ph, u_j) - \Phi(u_i, u_j+h) + \Phi(u_i, u_j) \geqq 0. \tag{6}$$

Treating now the second argument in (6) in the same way we obtain, for positive integers p, q and $h_i=ph$, $h_j=qh$,

$$\Phi(u_i+h_i, u_j+h_j) - \Phi(u_i+h_i, u_j) - \Phi(u_i, u_j+h_j) + \Phi(u_i, u_j) \geqq 0. \tag{7}$$

An appeal to the continuity of Φ establishes (7) for arbitrary h_i, $h_j \geqq 0$.

To prove (5), let I' be the group consisting of I and the index k, which belongs neither to I nor to J. Then

$$\begin{aligned}\Phi(U_{I'}+H_{I'}, U_J+H_J) &- \Phi(U_{I'}+H_{I'}, U_J) - \Phi(U_{I'}, U_J+H_J) + \Phi(U_{I'}, U_J) \\ &= \{\Phi(U_I+H_I, u_k+h_k, U_J+H_J) - \Phi(U_I+H_I, u_k+h_k, U_J) \\ &\quad - \Phi(U_I, u_k+h_k, U_J+H_J) + \Phi(U_I, u_k+h_k, U_J)\} \\ &\quad + \{\Phi(U_I, u_k+h_k, U_J+H_J) - \Phi(U_I, u_k+h_k, U_J) \\ &\quad - \Phi(U_I, u_k, U_J+H_J) + \Phi(U_I, u_k, U_J)\}.\end{aligned} \tag{8}$$

Applying this relation we can, beginning with (7), prove (5) by induction with respect to the number of elements of I and J.

In the same way, we can generalize (4) to

$$\int_0^\delta \{\Phi(x-t, U_I+H_I) + \Phi(x+t, U_I) - \Phi(x+t, U_I+H_I) - \Phi(x-t, U_I)\}\,dt \geqq 0. \tag{9}$$

Replacing in identity (8) u_k by $x-t$, u_k+h_k by $x+t$, and combining (5) and (9),

we obtain finally

$$\int_0^\delta \{\Phi(x-t, U_I+H_I, U_J+H_J) - \Phi(x-t, U_I, U_J+H_J) - \Phi(x+t, U_I+H_I, U_J) + \Phi(x+t, U_I, U_J)\}dt \geqq 0. \tag{10}$$

We can now prove (2) under the assumption that each of the functions $f_k(x)$ is a step-function, constant on each of the intervals $((s-1)/p, s/p)$, $s=1, \cdots, p$. For $1 \leqq s < p$ we consider the following *elementary operation* which gives a new set of functions $\bar{f}_k(x)$. We put $\bar{f}_k(x)=f_k(x)$ outside of $((s-1)/p, (s+1)/p)$; on $((s-1)/p, (s+1)/p)$, $\bar{f}_k(x)$ is the decreasing rearrangement of $f_k(x)$ on this interval. If I consists of the indices k for which $f_k(x)$ increases on $((s-1)/p, (s+1)/p)$, J of the indices for which $f_k(x)$ decreases, u_k is the smaller, u_k+h_k the larger of the two values of $f_k(x)$, then (10) with $x=s/p$, $\delta=1/p$ is exactly the inequality

$$\int_0^1 \Phi(x, f_1, \cdots, f_n)dx \leqq \int_0^1 \Phi(x, \bar{f}_1, \cdots, \bar{f}_n)dx.$$

By a finite number of elementary operations we can transform $f_1, \cdots, f_n$ into $f_1^*, \cdots, f_n^*$. This proves (2) in our particular case. In the general case we consider sequences $f_1^{(p)}, \cdots, f_1^{(p)}$, $p=1, 2, \cdots$ of uniformly bounded step-functions of our type such that $f_k^{(p)}(x) \to f_k(x)$ almost everywhere and pass to the limit $p \to \infty$ in the relation (2) for the $f_k^{(p)}$. This gives (2) in full generality.

It remains to show that (3) is equivalent to (3a) and (4) to (4a), if Φ has continuous second derivatives. If (4) holds, then for any i, $0<x<1$, $u_k \geqq 0$, there are arbitrary small $t>0$ with

$$\Phi(x+t, u_i+t) - \Phi(x-t, u_i+t) - \Phi(x+t, u_i) + \Phi(x-t, u_i) \leqq 0.$$

Dividing by $2t^2$ and making $t \to 0$, we obtain (4a). Conversely, from (4a) we deduce a relation stronger than (4), namely

$$\Delta^2\Phi = \Phi(x+t, u_i+h) - \Phi(x+t, u_i) - \Phi(x, u_i+h) + \Phi(x, u_i) \leqq 0. \tag{4b}$$

For if (4b) does not hold, there is a $c>0$ and a rectangle $R=(x, x+t; u_i, u_i+h)$ with side lengths t, h for which $\Delta^2\Phi \geqq cht$. Subdividing R, we obtain a sequence of rectangles with the same property which converge to a point (x^0, u_i^0). Then

$$\frac{\partial^2\Phi}{\partial x^0 \partial u_i^0} = \lim \frac{\Delta^2\Phi}{ht} \geqq c > 0,$$

which contradicts (4a). In the same way we treat the pair of relations (3), (3a).

Examples. The inequality (2) holds if $\Phi(u_1, \cdots, u_n)=u_1 \cdots u_n$. It holds for $\Phi=F(u_1+u_2+\cdots u_n)$ if and only if $F(u)$ is convex, that is $F(u+2h)-2F(u+h)+F(u) \geqq 0$. For example, $F(u)=-\log u$ has this property. Writing (2) in this

case for sums instead of integrals, we obtain Ruderman's inequality [2, Theorem II]

$$\prod_{s=1}^{p}\sum_{k=1}^{n} a_{sk} \geqq \prod_{s=1}^{p}\sum_{k=1}^{n} a_{sk}^{*}, \tag{11}$$

where $a_{sk} \geqq 0$ and the a_{sk}^{*}, $s=1, \cdots, p$ are the a_{sk}, $s=1, \cdots, p$ arranged in order of decreasing magnitude.

References

1. G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge, 1934.
2. H. D. Ruderman, Two new inequalities, this MONTHLY, 59, 1952, pp. 29–32.
3. A. Zygmund, Trigonometrical series, Warsaw-Lwow, 1935.

ON SUMS INVOLVING BINOMIAL COEFFICIENTS*

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In some problems of algebra† we are led to consider sums of the form $\sum_{r\geq 0} k_r A(n,r,\nu)\, B(n,r,\nu)\, C(n,r,\nu,)$ where $A, B, C, \cdots$, are binomial coefficients, depending on ν and also on one or two other integral parameters, and where the summation proceeds up to the first value of ν, for which one of the factors vanishes. A certain number of such sums can be found in [1] and [3]. However, the sums computed in (4), do not seem to appear in the literature. They do not follow readily by the methods of [3], and a direct proof, or a proof by induction, seems rather difficult. In what follows, we give a simple proof of (4), using well-known properties of Legendre's polynomials $P_n(x)$ and of the hypergeometric function $F(a, b; c; x)$.

Let $P_n(x) = \sum_{r=0}^{n} a_r^{(n)} x^r$ be the nth Legendre polynomial, and let $P_n^{(r)}(x)$ be its rth derivative. Then, by Maclaurin's formula, $P_n(x) = \sum_{r=0}^{n} \{x^r P_n^{(r)}(0)/r!\}$ so that

$$P_n^{(r)}(0)/r! = a_r^{(n)}. \tag{1}$$

Here the values of $a_r^{(n)}$ are (see [2], p. 11)

$$a_r^{(n)} = (-1)^{(n-r)/2} 2^{-n} \binom{n+r}{n} \binom{n}{(n+r)/2} \quad \text{if} \quad n \equiv r \pmod 2 \tag{2}$$

$$= 0 \qquad \text{otherwise.}$$

It also is known (see [4], pp. 61–62) that‡ the hypergeometric function

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† *E.g.* the study of the algebraic irreducibility of Legendre's polynomials in the field of rational numbers.

‡ The idea of this proof is due to Professor E. D. Rainville, who kindly suggested it to me in a letter.

$F(a, b; c; x) = 1 + \sum_{s=1}^{\infty} \frac{(a)_s (b)_s}{(c)_s} x^s$ satisfies $P_n(x) = F(-n, n+1; 1; (1-x)/2)$.

Here $(m)_s$ stands for $m(m+1)\cdots(m+s-1)$.

From the formula for the rth derivative of the hypergeometric function,

$$\frac{d^r}{dx^r} F(a, b; c; x) = \frac{(a)_r (b)_r}{(c)_r} F(a+r, b+r; c+r; x),$$

it follows that

$$P_n^{(r)}(x) = \frac{(n+r)!}{(n-r)!\, r!\, 2^r} F(-n+r, n+1+r; 1+r; (1-x)/2),$$

and, in particular,

$$P_n^{(r)}(0) = \frac{(n+r)!}{(n-r)!\, r!\, 2^r} \sum_{\nu=0}^{n-r} (-2)^{-\nu} \frac{(n-r)!}{\nu!(n-r-\nu)!} \frac{(n+r+\nu)!\, r!}{(n+r)!(r+\nu)!}.$$

This can be written in any of the following ways:

$$\begin{aligned} 2^r P_n^{(r)}(0)/r! &= \binom{n}{r} \sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n-r}{\nu} \binom{n+r+\nu}{n} \\ &= \binom{n+r}{r} \sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{\nu} \qquad (3) \\ &= \sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{n} \binom{r+\nu}{r}. \end{aligned}$$

If we substitute now in (1) the values of $a_r^{(n)}$ from (2) and also, successively, the three values of $P_n^{(r)}(0)$ from (3), after some obvious simplifications we obtain, respectively, for $r \equiv n \pmod 2$,

$$\begin{aligned} &\sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n-r}{\nu} \binom{n+r+\nu}{n} \\ &\qquad = (-1)^{(n-r)/2} 2^{r-n} \binom{2n}{n} \binom{n}{(n+r)/2} \binom{2n}{n+r}^{-1} \\ &\sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{\nu} = (-1)^{(n-r)/2} 2^{r-n} \binom{n}{(n+r)/2} \qquad (4) \\ &\sum_{\nu=0}^{n-r} (-2)^{-\nu} \binom{n}{r+\nu} \binom{n+r+\nu}{n} \binom{r+\nu}{r} \\ &\qquad = (-1)^{(n-r)/2} 2^{r-n} \binom{n+r}{n} \binom{n}{(n+r)/2}. \end{aligned}$$

If $r \not\equiv n \pmod 2$ the sums vanish.

Bibliography

1. J. Hagen, Synopsis der höheren Mathematik, v. 1, F. L. Dames, Berlin (1891).
2. E. Heine, Theorie der Kugelfunktionen, G. Reimer, Berlin (1878).
3. E. Netto, Lehrbuch der Combinatorik, B. G. Teubner, Leipzig (1901).
4. G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ. v. XXIII (1939).

NOTE ON A FORMULA OF GROSSWALD

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Using properties of the Legendre polynomials and the hypergeometric function, Grosswald has proved* the formula

$$(1)\qquad \sum_{\nu=0}^{n-r}(-2)^{-\nu}\binom{n}{r+\nu}\binom{n+r+\nu}{\nu}=(-1)^k 2^{-2k}\binom{n}{k},$$

where $n-r=2k$. We wish to point out that (1) is easily obtained by means of the known formula (W. N. Bailey, Generalized hypergeometric series, Cambridge, 1935, p. 11, formula (2)):

$$(2)\qquad F(a,b;\tfrac{1}{2}(a+b+1);\tfrac{1}{2})=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)}.$$

Indeed the left member of (1) evidently

$$\begin{aligned}
&=\binom{n}{n-r}\sum_{\nu=0}^{n-r}(-2)^{-\nu}\frac{(n-r)!r!}{(n-r-\nu)!(r+\nu)!}\binom{n+r+\nu}{\nu}\\
&=\binom{n}{n-r}\sum_{\nu=0}^{n-r}\frac{(r-n)_\nu(n+r+1)_\nu}{\nu!(r+1)_\nu}(\tfrac{1}{2})^\nu\\
&=\binom{n}{n-r}F(r-n,n+r+1;r+1;\tfrac{1}{2}),
\end{aligned}$$

and using (2) this becomes

$$\begin{aligned}
&\binom{n}{n-r}\frac{\Gamma(\frac{1}{2})\Gamma(r+1)}{\Gamma(\frac{1}{2}-k)\Gamma(r+k+1)}\\
&\quad=\frac{n!}{(n-r)!(r+k)!}(\tfrac{1}{2}-k)_k\\
&\quad=\frac{n!}{(2k)!(n-k)!}(-1)^k 2^{-k}(2k-1)(2k-3)\cdots 1\\
&\quad=(-1)^k 2^{-2k}\binom{n}{k}.
\end{aligned}$$

* This Monthly, vol. 60, p. 179.

CLASSROOM NOTES

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POPULATIONS WITH MEANS AND STANDARD DEVIATIONS OF SAMPLES INDEPENDENTLY DISTRIBUTED

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1. Introduction. Deming in his recent book [1] speaking of the joint distribution of the mean, $\bar{x}$, and standard deviation, s, of samples of n says, "These estimates $(\bar{x}, s)$ are independent in normal theory and nearly independent in some nonnormal theory, although no proof can be given by the author." Geary [2] was the first to prove that a necessary and sufficient condition for the normality of the parent distribution is that the sampling distributions of the mean and of the variance be independent. Of course suitable restrictions must be imposed on the parent distributions with regard to range, continuity, and differentiability. Geary used R. A. Fisher's general formulae for seminvariants. Another proof was given by Lukacs [3] using characteristic functions. The purpose of this note is to present a proof that makes use of only a few simple notions of function theory without appeal to the more sophisticated mathematical statistical concepts of seminvariants and characteristic functions.

2. A functional equation. A. T. Craig [4] has shown that if the parent population is $f(x)$, $-\infty < x < \infty$, then the joint distribution of $\bar{x}$ and s for samples of two is

$$F(\bar{x}, s) = 4f(\bar{x} - s)f(\bar{x} + s). \tag{2.1}$$

It is assumed that our functions are sufficiently well behaved as far as continuity and differentiability are concerned. If the mean and standard deviation are independent in general, they must be independent for samples of two. Thus a necessary condition for independence can be stated as

$$4f(\bar{x} - s)f(\bar{x} + s) = g(\bar{x})h(s). \tag{2.2}$$

Put $s = 0$ and we obtain

$$4[f(\bar{x})]^2 = h(0)g(\bar{x}). \tag{2.3}$$

Put $\bar{x} = 0$ and we obtain

$$4f(-s)f(s) = g(0)h(s). \tag{2.4}$$

Hence

$$4[(f(\bar{x})]^2 f(-s)f(s) = g(0)h(0)f(\bar{x} - s)f(\bar{x} + s) \tag{2.5}$$

which gives a condition on $f(x)$, the sampled population, so that the means and standard deviations of samples of two shall be independent.

We can now state the following theorem.

THEOREM 2.1. *Let a population be represented by* $f(x)$, $-\infty < x < \infty$, $f(x) > 0$, *then a necessary condition that the means* $\bar{x}$, *and standard deviations* s, *of samples are distributed independently is*

$$[f(\bar{x})]^2 f(-s)f(s) = kf(\bar{x} - s)f(\bar{x} + s) \tag{2.6}$$

where k *is a constant.*

3. Solution of the functional equation (2.6). It will now be shown that the only analytical function which satisfies the functional equation (2.6) is the normal curve. This proof was supplied by Professor C. A. Hayes, Jr. of the Mathematics Department, University of California at Davis. Take the logarithm of (2.6) and we have

$$2u(\bar{x}) + u(s) + u(-s) = \log k + u(\bar{x} + s) + u(\bar{x} - s) \tag{3.1}$$

where $u(\bar{x}) = \log f(\bar{x})$ and similarly for the other terms. Assume that derivatives exist, differentiate (3.1) with respect to $\bar{x}$, and obtain

$$2u'(\bar{x}) = u'(\bar{x} + s) + u'(\bar{x} - s). \tag{3.2}$$

Put $\bar{x} = s$ and get

$$2u'(\bar{x}) = u'(2\bar{x}) + u'(0). \tag{3.3}$$

Assume that $u(x)$ has a power series representation, substitute in (3.3), equate coefficients on both sides, and we find that, necessarily,

$$u(x) = u_0 + u_1 x + u_2 x^2 \tag{3.4}$$

or

$$f(x) = \exp(u_0 + u_1 x + u_2 x^2) \tag{3.5}$$

where the additional requirement that $f(x)$ be a distribution function requires that u_2 be negative. The linearity of $u'(x)$ can be established more generally from (3.2).

Thus it follows that the only population that is similar to the normal population which has means and standard deviations of samples independently distributed is the normal distribution.

References

1. W. E. Deming, Some theory of sampling, John Wiley and Sons, 1950, pp. 545–546.
2. R. C. Geary, Distribution of Student's ratio for nonnormal samples, J. R. S. Suppl. 3 (1936).
3. E. Lukacs, A characterization of the normal distribution. Annals of Math. Stat., v. 13, 1942, pp. 91–97.
4. A. T. Craig, The simultaneous distribution of mean and standard deviation in small samples, Annals of Math. Stat., vol. 3, 1932, pp. 126–140.

ON THE QUADRATIC FACTORS OF A POLYNOMIAL OF THE FOURTH DEGREE

J. G. CAMPBELL, University of Kentucky

Let $f(x)=x^4+ax^3+bx^2+cx+d$, where a, b, c and d are integers and suppose that $f(x)$ is the product of two factors of the form x^2+Ax+B, where A and B are integers. The following theorem may be used to isolate the coefficients A and B.

THEOREM. *If m is any positive or negative integer or zero, and m_i represent the j divisors of $f(m)$, $i=1, 2, \cdots, j$, then the set of numbers $(m_1-m^2, m_2-m^2, \cdots, m_j-m^2)$* includes $Am+B$.

Proof. Let $x^4+ax^3+bx^2+cx+d=(x^2+Ax+B)\ (x^2+A_1x+B_1)$. Then $f(m)=(m^2+Am+B)(m^2+A_1m+B_1)$. Thus (m^2+Am+B) is one of the divisors of $f(m)$ and $(m^2+Am+B)-m^2=Am+B$. Hence $Am+B$ is in the given set of numbers.

If n is any other integer, and n_h represent the k divisors of $f(n)$, $h=1, 2, \cdots, k$, then the set $(n_1-n^2, n_2-n^2, \cdots, n_k-n^2)$ includes $An+B$.

Solving the system

$$mA + B = m_i - m^2$$
$$nA + B = n_h - n^2,$$

we get

$$A = \frac{m_i - n_h}{m - n} - (m + n) \qquad i = 1, 2, \cdots, j,$$

$$B = \frac{mn_h - nm_i}{m - n} + mn \qquad h = 1, 2, \cdots, k.$$

Direct substitution of the divisors m_i and n_h yield the desired integers. Note that A_1 and B_1 are also determined by the above equations. By equating coefficients of the product of the quadratic factors to the coefficients of $f(x)$, we have

$$A + A_1 = a, \qquad BB_1 = d, \qquad B + B_1 + AA_1 = b, \qquad AB_1 + A_1B = c.$$

These relations may be used to further restrict the integers A and B. We select only pairs of values of the A's whose sum is equal to a and only pairs of values of the B's whose product is equal to d. These pairs of values are further restricted by testing them in $B+B_1+AA_1=b$ and $AB_1+A_1B=c$.

A numerical example is used to illustrate the method.* Find the roots of $x^4-2x^3-5x^2+10x-3=0$. Here $a=-2$, $b=-5$, $c=10$, $d=-3$. Set $m=0$ and $n=1$, and note that $f(0)=-3, f(1)=1$. The divisors m_i are, $(1, -1, 3, -3)$. The divisors n_h are, $(1, -1)$. The values of A and B can now be found from

* N. B. Conkwright, Introduction to the Theory of Equations, problem no. 5, p. 80.

$A=-(m_i-n_k)-1$ and $B=m_i$. Computing the A's we have $(-1, -3, 1, -5, 3)$. Selecting pairs of values whose sum is -2, we get $(1, -3)$ and $(-5, 3)$. Computing the B's, we get, $(1, -1, 3, -3)$. Selecting pairs of the B's whose product is -3, we have $(1, -3)$ and $(-1, 3)$. Substituting the B's in $B+B_1+AA_1=b$, we have $AA_1=-3$ or -7. The pair of A's satisfying these relations is $(1, -3)$; the pair $(-5, 3)$ is discarded. Let $A=1$, $A_1=-3$, then $B_1-3B=10$. The only pair of values satisfying this equation is $B_1=1$, $B=-3$. Thus, our problem reduces to solving

$$(x^2 + x - 3)(x^2 - 3x + 1) = 0.$$

The roots are, $(-1+\sqrt{13})/2$, $(-1-\sqrt{13})/2$, $(3+\sqrt{5})/2$ and $(3-\sqrt{5})/2$.

If $f(m)$ and $f(n)$ present few divisors, the labor incident to computing a root by this method is not greater than that of finding a rational root of the resolvent cubic equation.

SQUARE ROOTS IN GROUPS

W. R. Utz, University of Missouri

In beginning the study of finite groups a student soon observes that if a group is commutative, then the elements off the principal diagonal of the group table are symmetric with respect to the diagonal and any given element occurs in the table a number of times equal to the order of the group, from which it easily follows that the elements of the group occur exactly once in the diagonal when, and only when, the order of the group is odd. An equivalent way of stating this is that each element of a finite commutative group has a unique square root (*i.e.*, the equation $x^2=b$ has a unique solution for each element b of the group) when, and only when, the group is of odd order. That this is a property of all finite groups is not as easily established but it is possible to give a proof sufficiently elementary to be understood early in the study of groups and, in most classes, there will be some students who can provide the proof themselves when the theorem is suggested.

We first observe that the "perfect squares" in the group appear on the diagonal of the group table. Thus if any element appears twice on the diagonal, then some element of the group is omitted from the diagonal and such an element does not have a square root. This proves the following lemma.

Lemma. *If G is a finite group and each $a\epsilon G$ has a square root, then the root is unique.*

Theorem 1. *Each element of a finite group has a square root if, and only if, the order of the group is odd.*

Proof. Suppose that G is a finite group of even order. Some element of G has period 2. To see this, write $G=S\cup R$ where S is the set of all elements of G of period greater than 2. If $x\epsilon S$, then $x^{-1}\epsilon S$ and $x\neq x^{-1}$. Thus S contains an even

number of elements of G. Since S has an even number of elements of the even ordered group G, R contains the identity and an odd number of elements of period 2 hence at least one element of period 2.

Now suppose that G is of even order and that each element of G has a square root. From the previous paragraph there exists $a \epsilon G$, $a \neq i$, where i denotes G's identity, such that $a^2 = i$. This is contrary to the Lemma since i has at least two square roots i and a.

To prove the other half of the theorem, let G be a group of odd order $n = 2k + 1$. Then $1 = n - 2k$ and since $b^n = i$ for each b in G,

$$b = b^{n-2k} = b^{-2k} = (b^{-k})^2 = c^2,$$

where $c = b^{-k} = (b^{-1})^k$. For each b, the element c is uniquely determined and qualifies as a square root of b. (The author is indebted to the referee for shortening this proof.)

The theorem given above is a special case of the following theorem.

THEOREM 2. *If G is a finite group and r is a positive integer, then each element of G has an r-th root when, and only when, r is prime to the order of G.*

The "only when" half of this theorem is a consequence of the corollary to Sylow's Theorem which states: If a prime number p is a factor of the order of a finite group G, then G contains an element of period p. The other half is readily established as in Theorem 1, for if r is prime to n, the order of G, then there exist integers, h and k, such that $1 = hr + kn$.

The theorems given above, in one form or another, may be known to many algebraists. However, the author does not know a comparable theorem for infinite groups. It seems that even a sufficient condition that in a group, at least as general as the group of all non-singular $n \times n$ matrices over the complex field, the equation $x^2 = b$, for each b of the group, have a solution would be of interest since for the matrix group this is commonly established by special methods that are not group-theoretic (see, for example, Bôcher, Higher Algebra, p. 299).

ON AN ELEMENTARY DERIVATION OF CRAMER'S RULE

D. E. WHITFORD and M. S. KLAMKIN, Polytechnic Institute of Brooklyn

The purpose of this note is to point out an elementary derivation of Cramer's rule which should be easily understood by freshman students.

Consider the simultaneous set of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{1}$$

Now

$$x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

by the elementary transformations of a determinant. Hence if x is to satisfy equations (1) it is necessary that

$$x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

or

$$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \Delta, \quad \text{provided} \quad \Delta \neq 0,$$

where Δ is the determinant of the coefficient matrix of the system (1). Similarly

$$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \Delta, \qquad z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \Delta.$$

That these conditions are sufficient, when $\Delta \neq 0$, can be established by substituting back into (1), which gives

$$a_r\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} + b_r\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} + c_r\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = d_r\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

That this is true follows from

$$\begin{vmatrix} a_r & b_r & c_r & d_r \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0$$

since the top row is equivalent to one of the other rows.

The method can be extended immediately to n linear equations in n unknowns.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1056. *Proposed by W. R. Ransom, Tufts College*

A rectangular room has a spider one foot down from the ceiling at the middle of one end, and a fly one foot up from the floor at the middle of the other end. There are three paths by which the spider can crawl to the fly, and which become straight lines when the sides of the room are properly developed. What dimensions of the room will make these three paths equal in length?

E 1057. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find the sum of the first n terms of the series

$$\sec\theta + (\sec\theta \sec 2\theta)/2 + (\sec\theta \sec 2\theta \sec 4\theta)/4 + \cdots .$$

E 1058. *Proposed by M. Perisastri, M. R. College, Vizianagram, India*

If $d_1, d_2, \cdots, d_k$ are the divisors of n, show that

$$(d_1 d_2 \cdots d_k)^2 = n^k, \tag{1}$$

$$d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k} = n^{d_1+d_2+\cdots+d_k} e^{-nrk}, \qquad 0 \leqq r \leqq (\log 3)/3. \tag{2}$$

E 1059. *Proposed by Chih-yi Wang, Hampton Institute*

Let a circle and an inscribed closed polygon of n sides be given. Show that the product of the distances of a point on the circumference of the circle from the sides of the polygon is equal to the product of the distances of the same point from the sides of the tangential polygon (*i.e.*, the polygon formed by the tangents to the circle at the vertices) of the given polygon.

E 1060. *Proposed by H. K. Crowder, Case Institute of Technology*

Let D be a determinant of order n whose ith row, $i=1, \cdots, n$, is

$$a_i, a_{i-1}, \cdots, a_1, 1, 0, \cdots, 0.$$

Show that

$$D = (-1)^{n\Sigma} \frac{(-1)(-2)\cdots(-\Sigma\alpha_i)}{\alpha_1!\alpha_2!\cdots\alpha_n!} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n},$$

where the sum is taken over all non-negative integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n.$$

SOLUTIONS

Multiplications Involving Only Two Digits

E 1012 [1952, 249]. *Proposed by Oystein Ore, Yale University*

In a multiplication like

$$\begin{array}{r} 11 \\ \times 11 \\ \hline 11 \\ \underline{11} \\ 121 \end{array}$$

there occur only the two digits 1 and 2. Find all multiplications involving exactly two digits.

Solution by the Proposer. The two numbers to be multiplied shall be denoted by A and B. Each contains at most two different digits a and b. We shall divide the problem into simpler cases.

(I) A and B have only a single digit each.

By looking at an ordinary multiplication table one finds that there are only the cases

$$0 \times a = a \times 0 = 0, \qquad 1 \times a = a \times 1 = a,$$
$$2 \times 2 = 4, \qquad 3 \times 3 = 9, \qquad 5 \times 5 = 25, \qquad 6 \times 6 = 36,$$

involving only two digits.

(II) $A = a \neq 0$ has a single digit, B has more than one digit.

Through the use of (I) it is verified that there are only the solutions

$$2 \times (22 \cdots 2), \qquad 3 \times (33 \cdots 3), \qquad 1 \times (bb \cdots b), \qquad a \times (11 \cdots 1).$$

(III) A has more than one digit, $B = b$ is a single digit.

This is the most involved case and it seems necessary to examine the various possibilities for b separately. The results are, except for $b = 0$ and $b = 1$,

$$(22 \cdots 2) \times 2, \qquad (33 \cdots 3) \times 3,$$
$$(11 \cdots 1) \times b \text{ arbitrary}, \qquad 86 \times 8.$$

(IV) A and B contain at least two digits each.

Each line in a multiplication is then of the type given in (III) and it is readily verified that the only permissible digits can be 0 and 1. If all digits are unity the only solution is found to be

$$(11 \cdots 1) \times 11.$$

Thus this is the only solution to our problem for at least two digits in A and B, with zero not permitted as a digit. If zero is permitted the result is not so simple to formulate.

A Code Problem

E 1026 [1952, 465]. *Proposed by Michael Golomb, Purdue University*

Orders have been given in a certain country that from now on all scientific writing shall be done in a secret transcription, consisting of a permutation of the alphabet of the country. A foreigner, unacquainted with the language and its alphabet is given the privilege of seeing the transcribed form of any five letters he may designate. He wishes to learn the common form of the word *atom*, which he is told has five letters. How can he manage this, if reference to the inverse transcription is ruled out?

Solution by the Proposer. Let x be the word for atom, $c_1(x)$ the transcribed form of x, and $c_k(x) = c_1(c_{k-1}(x))$ for $k = 2, 3, \cdots$. The foreigner asks: "What is the transcribed form of $c_{p-1}(x)$ where $p = n!$ (or $p =$ l.c.m. of numbers $2, 3, \cdots, n$) and n is the number of letters in your alphabet." It is clear that $c_p(x) = x$.

Also solved by C. S. Ogilvy.

Total Area of a Set of Circles

E 1027 [1952, 465]. *Proposed by Ruth Clark and Robert Oeder, Los Alamos Scientific Laboratory*

Find the total area enclosed by the set of circles formed as follows: Construct the inscribed circle of a triangle ABC and draw the tangent to this circle which is parallel to a selected side of the triangle; construct the inscribed circle of the new triangle so cut off and draw the tangent to this circle which is parallel to the selected side of triangle ABC; repeat the process indefinitely. Also show that the set of maximum area is obtained if the lines are drawn parallel to the shortest side of triangle ABC.

Solution by Julian Braun, Washington, D. C. Let K, s, r_0, r_n denote the area, semiperimeter, radius of the excircle on the selected side a, and radius of the nth constructed incircle, of the given triangle. Then

$$r_0 = K/(s - a), \qquad r_1 = K/s,$$

whence

$$r_1/r_0 = r_{n+1}/r_n = 1 - a/s.$$

This ratio is largest, and thus the required area a maximum, if a is the shortest side of the triangle. The total area of the set of incircles is

$$\sum_{i=1}^{\infty} \pi r_i^2 = \pi(K/s)^2 \sum_{i=0}^{\infty} (1 - a/s)^{2i}$$

$$= \pi K^2/(2as - a^2) = \pi K^2/a(b + c).$$

Also solved by Leon Bankoff (in two ways), D. H. Browne, C. V. Fronabarger, Harry Furstenberg, Arthur Gregory, Vern Hoggatt, H. I. James, M. S.

Klamkin, A. E. Livingston, B. Martin and J. C. Wu (jointly), C. S. Ogilvy, F. D. Parker, W. O. Pennell, R. R. Phelps, L. A. Ringenberg, W. J. Robinson, Azriel Rosenfeld, K. Subba Rao, C. W. Trigg, J. V. Whittaker, and the proposers. Late solutions by C. L. Gape and D. E. Konhauser.

Pythagorean Triangles and Fibonacci Numbers

E 1028 [1952, 465]. *Proposed by Vern Hoggatt, Oregon State College*

Do there exist Pythagorean triangles whose sides are Fibonacci numbers?

Note by Norman Miller, Queen's University. This problem was assigned as an exercise to my freshman class in algebra and a number of answers were received which made use of one or other of the following arguments:

(1) No three numbers of a Fibonacci sequence can represent the sides of *any* triangle, since of any three chosen the sum of the two smaller is not greater than the third;

(2) If three numbers from a Fibonacci sequence, in order of magnitude, are a, b, and c, then $c \geqq a+b$ and $c^2 > a^2+b^2$.

Remarks by W. F. Cheney, Jr., University of Connecticut. It is quite likely that (3, 4, 5) and (5, 12, 13) are the only Pythagorean triangles *two* of whose sides are numbers in the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \cdots .$$

If we refer to these Fibonacci numbers by their position numbers in the sequence, those numbered $6n$ will always be an even leg of a Pythagorean triangle; those numbered $6n \pm 1$, $6n \pm 2$ will always be an odd leg of some Pythagorean triangle; those numbered $6n \pm 1$ will always be the hypotenuse of some Pythagorean triangle; some, but not all, of those numbered $6n+2$ may be a hypotenuse; none numbered $6n+3$ can constitute any side of a Pythagorean triangle; those numbered $6n+4$ can never be a hypotenuse.

Also solved by A. P. Boblétt, Arthur Gregory, P. G. Kirmser, M. S. Klamkin, Sam Kravitz, R. J. Mercer, Leo Moser, J. V. Pennington, L. L. Pennisi, L. A. Ringenberg, Azriel Rosenfeld, K. Subba Rao, J. V. Whittaker, Margaret Willerding, and the proposer.

E 854 Again

E 1029 [1952, 465]. *Proposed by A. E. Currier, United States Naval Academy*

Sum the series $1+1/3+2!/(3)(5)+3!/(3)(5)(7)+ \cdots$.

Solution by K. Subba Rao, M. R. College, Vizianagram, India. We have, in the interval $-1<x<1$,

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \cdots$$

$$= x\left[1 + \frac{1}{3}2x^2 + \frac{2!}{3\cdot 5}2^2x^4 + \frac{3!}{3\cdot 5\cdot 7}2^3x^6 + \cdots\right].$$

Putting $x=1/\sqrt{2}$ in each member, and simplifying, we have

$$1 + 1/3 + 2!/(3)(5) + 3!/(3)(5)(7) + \cdots = \pi/2.$$

Also solved by Julian Braun, L. Carlitz, Richard Courter, H. E. Fettis, Harry Furstenberg, H. W. Gould, N. G. Gunderson, Vern Hoggatt, A. S. Howard, P. G. Kirmser, M. S. Klamkin, W. D. Lambert, A. E. Livingston, Norman Miller, F. R. Olson, W. O. Pennell, L. L. Pennisi, M. R. Spiegel, O. E. Stanaitis, C. A. Swanson, Chih-yi Wang, J. V. Whittaker, R. E. Wild, and the proposer.

Editorial Note. It has been pointed out that this problem is the same as E 854 [1949, 633–35]. There five different solutions were published, and three references given.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4528. *Proposed by Paul Erdös, National Bureau of Standards, Los Angeles*

Consider sequences of consecutive positive integers, n, $n+1$, $\cdots$, $n+k$, which have the property that one of the integers is relatively prime to all the others. (1) Prove that, if one of the integers is a prime, the property holds. (2) Show that sequences exist which do not have the property.

4529. *Proposed by C. D. Olds, San Jose State College, California*

Can one find a convergent series $a_0+a_1x+a_2x^2+\cdots$, where the a's are real and positive and such that all the roots of each of the equations

$$a_0 + a_1x + a_2x^2 = 0$$

$$a_0 + a_1x + a_2x^2 + a_3x^3 = 0$$

$$\cdots\cdots\cdots\cdots$$

are real?

4530. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$, let A', B', C', D' be the feet of the altitudes AA', BB', CC', DD'. The planes drawn through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ perpendicular to BC, CA, AB, DA, DB, DC respectively, are concurrent at a point P, which is the radical center of the spheres described with the vertices A, B, C, D as centers and with the altitudes AA', BB', CC', DD' as radii.

4531. *Proposed by R. M. Redheffer, University of California, Los Angeles*

Let $f(z)=\sum a_n z^n$ have simple poles with positive residues at the points $d_m \neq 0$, and similarly for $g(z)=\sum b_n z^n$ at the points $e_m \neq 0$. Both functions are supposed regular elsewhere. Then the function $\sum a_n b_n z^n$ has simple poles with negative residues at the points $d_m e_n$ and no other finite singularities.

4532. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime $\geqq 3$ and let g denote a primitive root (mod p). Prove

1. $$1+\sum_{r=1}^{(p-3)/2}\frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)}\equiv\prod_{r=1}^{(p-3)/2}(1+g^r)\pmod{p}.$$

2. If $p\equiv 3 \pmod 4$, then

$$1+\sum_{r=1}^{(p-3)/2} g^{r(r+1)/2}\frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)}\equiv\prod_{r=1}^{(p-3)/4}(1+g^{2r})\pmod{p};$$

If $p\equiv 1 \pmod 4$, the sum vanishes.

SOLUTIONS

A Definite Integral

4469 [1952, 45]. *Proposed by A. J. Coleman, University of Toronto*

Evaluate the integral

$$\int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{x^2(1+x^2)}\,dx.$$

The dominant term when a is very much smaller than b is also desired. (The integral was met in a discussion of the diffusion of gamma rays.)

Solution by the Proposer. Since $1/x^2(1+x^2)=x^{-2}-(1+x^2)^{-1}$, we have

$$I(a, b) = J(a, b) - K(a, b),$$

where $I(a, b)$ is the given integral and

(1) $$J(a, b) = \int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{x^2}\,dx,$$

$$K(a, b) = \int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{1+x^2}\,dx. \tag{2}$$

The substitution $x = z^{-1}$ reduces (1) to a known integral*

$$J(a, b) = \int_0^\infty e^{-b^2z^2-a^2/z^2}dz = \frac{\sqrt{\pi}}{2b}\,e^{-2ab}. \tag{3}$$

To evaluate K, we note that the substitution $x = y^{-1}$ gives

$$K(a, b) = K(b, a) = K. \tag{4}$$

Further

$$K - \frac{1}{2a}\frac{\partial K}{\partial a} = \int_0^\infty \frac{e^{-a^2x^2-b^2/x^2}}{1+x^2}\,dx + \int_0^\infty \frac{x^2e^{-a^2x^2-b^2/x}}{1+x^2}\,dx$$

$$= \int_0^\infty e^{-a^2x^2-b^2/x^2}dx = J(b, a) = \frac{\sqrt{\pi}}{2a}\,e^{-2ab}$$

by (3). Thus we obtain

$$\frac{\partial}{\partial a}\left(e^{-a^2}K\right) = -\sqrt{\pi}\,e^{-a^2-2ab},$$

$$e^{-a^2}K(a, b) = f(b) - \sqrt{\pi}\int_0^a e^{-x^2-2bx}dx,$$

where $f(b)$ does not involve a. It follows that

$$\begin{aligned} e^{-a^2-b^2}K(a, b) &= F(b) - \sqrt{\pi}\int_0^a e^{-(x+b)^2}dx \\ &= F(a) - \sqrt{\pi}\int_0^b e^{-(x+a)^2}dx, \end{aligned} \tag{5}$$

where we have made use of (4), and $F(b) = e^{-b^2}f(b)$. Putting $a = 0$ in (5) and then $b = 0$ in the result, we have

$$F(b) = e^{-b^2}K(0, b) = F(0) - \sqrt{\pi}\int_0^b e^{-x^2}dx,$$

$$F(0) = K(0, 0) = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

We continue as follows:

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* Goursat-Hedrick, *Mathematical Analysis*, v. I, p. 373. In exercise 3, replace x by bz and a by a/b.

$$e^{-a^2-b^2}K(a,b) = \frac{\pi}{2} - \sqrt{\pi}\int_0^b e^{-x^2}dx - \sqrt{\pi}\int_0^a e^{-(x+b)^2}dx$$

$$= \frac{\pi}{2} - \sqrt{\pi}\int_0^{a+b} e^{-x^2}dx = \sqrt{\pi}\left[\int_0^\infty e^{-x^2}dx - \int_0^{a+b} e^{-x^2}dx\right]$$

$$= \sqrt{\pi}\int_{a+b}^\infty e^{-x^2}dx.$$

$$K(a,b) = \sqrt{\pi}\, e^{a^2+b^2}\int_{a+b}^\infty e^{-x^2}dx.$$

Thus we have finally

$$I(a,b) = \sqrt{\pi}\, e^{-2ab}\left[\frac{1}{2b} - e^{(a+b)^2}\int_{a+b}^\infty e^{-x^2}dx\right]. \tag{6}$$

The integral involved is the well known, and much tabulated, remainder of the Error Function. Since†

$$e^{y^2}\int_y^\infty e^{-x^2}dx = \frac{1}{2y}\left[1 - \frac{1}{2y^2} + \frac{1\cdot 3}{2^2y^4} - \frac{1\cdot 3\cdot 5}{2^3y^6} + \cdots\right],$$

we have, for large values of $a+b$,

$$I(a,b) = \sqrt{\pi}\, e^{-2ab}\left[\frac{1}{2b} - \frac{1}{2(a+b)} + \frac{1}{4(a+b)^3} - \cdots\right]. \tag{7}$$

For small a/b, the dominant term of this will be

$$I(a,b) \sim \frac{\sqrt{\pi}}{2}\,\frac{a}{b^2}\, e^{-2ab}. \tag{8}$$

Throughout the above proof the differentiation under the integral sign is readily justified by the powerful convergence of e^{-x^2}.

An Integral Inequality

4471 [1952, 46]. *Proposed by Ky Fan, University of Notre Dame*

Let $K(x, y)$ be a non-negative Lebesgue integrable function over the square $a \leqq x \leqq b$, $a \leqq y \leqq b$. Suppose that B is a positive constant such that $\int_a^b K(x, y)dy \leqq B$ for almost all x in $[a, b]$, and also $\int_a^b K(x, y)dx \leqq B$ for almost all y in $[a, b]$. If two finite-valued functions $f(x)$, $g(x)$ are both non-negative and non-increasing in $[a, b]$, prove that

$$\int_a^b \int_a^b K(x, y)f(x)g(y)dxdy \leqq B\int_a^b f(x)g(x)dx. \tag{1}$$

† Jahnke and Emde, *Tables of Functions*, Dover, 1943, p. 24

[For the particular case when $K(x, y)$ is constant, (1) is known as Tchebychef's inequality. See Hardy, Littlewood, and Polya, *Inequalities*, Cambridge 1934, p. 168, Theorem 236.]

Solution by the Proposer. Obviously we may assume $B=1$. Let

$$h(x) = \int_a^b K(x, y)g(y)dy.$$

Then (1) can be written

$$\int_a^b f(x)[h(x) - g(x)]dx \leqq 0. \tag{2}$$

Consider an arbitrary point c in $[a, b]$. Let

$$k(y) = \int_a^c K(x, y)dx.$$

Then

$$k(y) \geqq 0 \text{ in } [a, b]; \qquad k(y) \leqq 1 \text{ for almost all } y \text{ in } [a, b]; \tag{3}$$

$$\int_a^b k(y)dy \leqq c - a; \tag{4}$$

and

$$\int_a^c h(x)dx = \int_a^b k(x)g(y)dy = \int_a^c k(y)g(y)dy + \int_c^b k(y)g(y)dy. \tag{5}$$

As $g(x)$ is non-increasing and non-negative, (3) and (4) imply

$$\int_a^b k(y)g(y)dy \leqq g(c)\int_c^b k(y)dy \leqq g(c)\left[c - a - \int_a^c k(y)dy\right].$$

Consequently we have, by (5):

$$\int_a^c h(x)dx \leqq \int_a^c k(y)[g(y) - g(c)]dy + (c - a)g(c)$$

and then, by the second part of (3):

$$\int_a^c h(x)dx \leqq \int_a^c [g(y) - g(c)]dy + (c - a)g(c) = \int_a^c g(y)dy.$$

Hence,

$$\int_a^c [h(x) - g(x)]dx \leqq 0 \qquad \text{for all } c \text{ in } [a, b]. \tag{6}$$

As $f(x)$ is finite-valued and non-increasing, we can apply the Second Mean Value Theorem. There exists a point c in $[a, b]$ such that

$$\int_a^b f(x)[h(x) - g(x)]dx = f(a)\int_a^c [h(x) - g(x)]dx + f(b)\int_c^b [h(x) - g(x)]dx.$$

Since $f(a) \geqq f(b)$, we have by (6):

$$f(a)\int_a^c [h(x) - g(x)]dx \leqq f(b)\int_a^c [h(x) - g(x)]dx$$

and therefore, since $f(b) \geqq 0$:

$$\int_a^b f(x)[h(x) - g(x)]dx \leqq f(b)\int_a^b [h(x) - g(x)]dx \leqq 0,$$

which is the desired inequality (2).

Editorial Note. H. J. Zimmerberg points out the similarity of the above inequality to Theorem 2.1 in W. T. Reid, Symmetrizable completely continuous linear transformations in Hilbert space, *Duke Mathematical Journal*, v. 18 (1951) pp. 41–56. The inequality is shown to hold also under different hypotheses. It does not appear to be a simple problem to derive one from the other.

A Problem in Plane Areas

4472 [1952, 46]. *Proposed by G. T. Williams, Elmont, New York*

If n is an odd integer >1, the curve $x^n+y^n=1$ has the line $x+y=0$ as asymptote, and the area in the second (or fourth) quadrant between the curve and its asymptote exists and is equal to the area in the first quadrant divided by $2\cos(\pi/n)$.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. The asymptote may be easily obtained by familiar methods.

Let us denote by A_1 the area in the first quadrant and by A_2 the area between the curve and its asymptote in the fourth quadrant. Then

$$A_1 = \int_0^1 (1 - x^n)^{1/n}dx, \qquad A_2 = \frac{1}{2} + \int_1^\infty [x + (1 - x^n)^{1/n}]dx.$$

Since $n \geqq 3$ and $x+(1-x^n)^{1/n} = O(1/x^{n-1})$, the second integral exists. Making the substitutions $x = (\sin t)^{2/n}$ and $x = (\sin t)^{-2/n}$ in A_1 and A_2 respectively, we obtain

$$A_1 = \frac{2}{n}\int_0^{\pi/2} (\sin t)^{2/n-1}(\cos t)^{2/n+1}dt,$$

$$A_2 = \frac{1}{2} + \frac{2}{n}\int_0^{\pi/2} [1 - (\cos t)^{2/n}](\sin t)^{-4/n-1}\cos t dt.$$

Integration by parts yields

$$A_2 = \frac{1}{n}\int_0^{\pi/2} (\sin t)^{1-4/n}(\cos t)^{2/n-1}dt.$$

These integrals are well known in terms of the Beta-function. Hence their ratio is found to be

$$\frac{A_1}{A_2} = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)} = \frac{\pi}{\sin\frac{\pi}{n}}\,\frac{\sin\frac{2\pi}{n}}{\pi} = 2\cos\frac{\pi}{n}$$

which gives the desired result immediately.

Also solved by G. P. Henderson, M. Perisastri, M. R. Spiegel, F. Underwood, R. E. Wild, and the Proposer.

A Theorem on Differences (mod p)

4473 [1952, 109]. *Proposed by J. B. Kelly, Institute for Advanced Study*

Show that the differences of order $(p+1)/2$ of the sequence

$$\left(\frac{1}{p}\right), \left(\frac{2}{p}\right), \cdots, \left(\frac{x}{p}\right), \cdots, \left(\frac{p-1}{p}\right)$$

are divisible by the prime p. Here (x/p) is the Legendre symbol. Show, conversely, that the only sequences, $\{f(x)\}$, of length $p-1$ consisting only of 1's and -1's, and having this property are $f(x)\equiv 1$, $f(x)\equiv -1$, $f(x)=(x/p)$, $f(x)=-(x/p)$, $(x=1, 2, \cdots, p-1)$.

Solution by Leonard Carlitz, Duke University. Since the rth differences of a polynomial of degree $<r$ vanish and since

$$\left(\frac{x}{p}\right) \equiv x^{(p-1)/2} \pmod{p},$$

it follows at once that

$$\Delta^r\left(\frac{x}{p}\right) \equiv 0 \pmod{p} \qquad \text{for } r \leqq (p+1)/2.$$

To prove the converse, we remark first that by the Lagrange interpolation a function defined on the set $1, 2, \cdots, p-1$ and with values (mod p) is given by

$$f(x) = -\sum_{a=1}^{p-1} af(a)\,\frac{x^{p-1}-1}{x-a}. \tag{1}$$

In other words, $f(x)$ can be represented by a polynomial. Now suppose the numbers $1, 2, \cdots, p-1$ separated into two sets A, B such that $f(a)=1$ for $a\in A$, $f(b)=-1$ for $b\in B$. Then (1) becomes

$$f(x) = -\sum_a a\frac{x^{p-1}-1}{x-a} + \sum_b b\frac{x^{p-1}-1}{x-b}. \tag{2}$$

In the next place the hypothesis $\Delta^{(p+1)/2}f(x)\equiv 0$ implies deg $f(x)\leqq(p-1)/2$. Accordingly (2) implies

$$\sum_a a^r = \sum_b b^r, \qquad (1\leqq r<(p-3)/2). \tag{3}$$

Put $g(x)=\prod_a(x-a)$, $h(x)=\prod_b(x-b)$, so that $x^{p-1}-1\equiv g(x)h(x)$. Now if deg $g(x)>$deg $h(x)$, it follows from (3) and the familiar congruence

$$\sum_{c=1}^{p-1} c^r \equiv 0 \pmod{p} \qquad (1\leqq r\leqq p-2)$$

that $h(x)$ reduces to a power of x; consequently $h(x)=1$ and $f(x)=1$. Similarly the supposition deg $g(x)<$deg $h(x)$ implies $g(x)=1$ and $f(x)=-1$. If then we take deg $g(x)=$deg $h(x)=(p-1)/2$, we see that (3) implies

$$g(x) - h(x) = 2C \text{ (constant)},$$
$$x^{p-1} - 1 = g^2(x) - 2Cg(x) = \{g(x) - C\}^2 - C^2,$$

so that $C=\pm 1$. It follows that

$$g(x) = x^{(p-1)/2} \pm 1, \qquad h(x) = x^{(p-1)/2} \mp 1. \tag{4}$$

Taking the lower signs in (4) we find that (2) reduces to $f(x)=x^{(p-1)/2}$, while the upper signs lead to $f(x)=-x^{(p-1)/2}$

Also solved by the Proposer.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

ON INTEGRAL COORDINATES

NORMAN ANNING, University of Michigan

In the classroom integral coordinates have some obvious uses. Obvious is a teacher's word for "as plain as the truck you overtake on the road (*ob viam*)."

The teacher who wants a circle or a parabola with integral points has no trouble in making the arrangements. Here are some handy ellipses.

Obvious solutions of the equation $x^2+3y^2=4(a^2+ab+b^2)$ are

$$\begin{array}{lccc} x^2 = (a-b)^2, & (2b+a)^2, & (2a+b)^2, \\ y^2 = (a+b)^2, & a^2, & b^2. \end{array}$$

Choose for a and b any positive integers and the stage is set. If at any time such a conic with its dozen integral points begins to seem too pat and regular, variety can be introduced by the transformations $x=Y+X$, $y=Y-X$. Thus $x^2+3y^2=52$ becomes $X^2-XY+Y^2=13$ and it will be found that the twelve integral points have gone over unharmed. Anyone who feels the need of more than a dozen integral points on one ellipse should try replacing 52 by 196, or 13 by 91, or 13 by 1729.

Only one application will be suggested: the verification of the Pascal Theorem can be brought within the range of the freshman.

Editorial Note to Students. Professor Anning's note suggests avenues of further research. Can you discover other "handy" curves, in particular conic sections? We shall be glad to receive any that you find.

NUMBER MAGIC

To guess a person's age: 1) Have him add 1953 to his age A, and then determine the digit-sum S of the total. Since $A \equiv S \bmod 9$, the person's age should be within the range of guesswork. For example, if $A \equiv 5 \bmod 9$, then $A = 5$, 14, 23, 32, 41, or 50 and one should be able to match the proper value of A and the individual. 2) Or, have him multiply his age by 1953, add his age to the product, and then determine the digit-sum of the total. Again, $A \equiv S \bmod 9$, and we proceed as in (1).

ON SHORT METHODS OF MULTIPLICATION

Many short methods of multiplication have been suggested, but most of them are applicable only to special situations. Here is a brief list, without explanations, of some of the more useful of these short methods.

1. To square numbers ending in 5.
   Rule. $(10a+5)^2=100a(a+1)+25$.
2. To square numbers near 25.
   Rule. $(25+x)^2=100x+(25-x)^2$.
3. To square numbers near 50.
   Rule. $(50+x)^2=100(25+x)+x^2$.
4. To square numbers near 100.
   Rule. $(100+x)^2=100(100+2x)+x^2$.
5. To find the product of two "teen" numbers.
   Rule. $(10+a)(10+b)=10(\overline{10+a}+b)+ab$.

6. To find the product of numbers having small complements.
   Rule. $(10-a)(10-b)=10(\overline{10-a-b})+ab$,
   $(100-a)(100-b)=100(\overline{100-a-b})+ab$, etc.
7. To find the product of numbers having small supplements.
   Rule. $(100+a)(100+b)=100(\overline{100+a+b})+ab$, etc.
8. Generalization of (6).
   Rule. $ab=(x-a)(x-b)+x(\overline{a+b-x})$.

We add to this list another method which does not seem to be as well known. The method is quite general in its application; indeed, it includes some of the other methods as special cases.

9. Rule. $ab=(a-c)(b+c)+(\overline{a-a-c})(\overline{b-a-c})$
   $=(a+c)(b-c)+(\overline{a+c}-a)(\overline{a+c}-b)$.

(Note that the second term on the right is the product of the differences between *two* factors of one set and *one* factor of the other set.)

Examples. $(27)(19)=(30)(16)+(3)(11)=480+33=513.$ $(c=3)$
$(52)(33)=(50)(35)+(2)(-17)=1750-34=1716.$ $(c=2)$
$(146)(86)=(150)(82)+(4)(64)=12300+256=12556.$ $(c=4)$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

An Introduction to the Theory of Differential Equations. By Walter Leighton. McGraw-Hill Book Company, 1952. viii+174 pages. $3.50.

This is a textbook for the mathematics major and other students interested in the theory of elementary differential equations. Emphasis is placed on a careful statement of the existence theorems for solutions of the various types of differential equations studied. The book is limited to the study of ordinary differential equations of the first degree.

The topics discussed in the order in which they appear are: the nature of solutions of differential equations with a statement of the fundamental existence theorems for equations of the first and second orders; solutions of first order equations with simple applications; the linear equation with variable coefficients including linear dependence and existence theorems with most of the work limited to second order equations; the method of variation of parameters; the linear equation with constant coefficients with applications chiefly from classical

mechanics; solutions by means of power series with a discussion of the existence of solutions and, for the second order equation, a discussion of singular points; systems of linear equations; the method of successive approximations; oscillation theory with Sturm's separation and comparison theorems as they apply to self-adjoint linear equations of the second order; characteristic functions and the problem of expanding a function in a series of functions orthogonal with respect to a weight function. Appendix 1 deals with a proof of the fundamental existence theorem for two simultaneous first order equations, Appendix 2 gives a proof of the implicit function theorem, Appendix 3 proves the existence of an integrating factor for the first order equation, and Appendix 4 is devoted to regular integrals.

Some of the topics are treated so briefly that one wonders why they were included. The author states in the preface that he has given more than the usual amount of attention to the method of successive approximations but the reviewer finds that only two pages of the text are devoted to this method as it applies to differential equations. This method is used in the appendix to prove the fundamental existence theorem. The work on functions orthogonal with respect to a weight function and the problem of the expansion of a function in a series of such functions is covered in four pages. On the other hand, in the chapter on oscillation theory, the self-adjoint differential equation of the second order is studied in some detail.

The author seems to have succeeded very well in putting before the student statements of the theorems relating to solutions of equations of the types studied. Since the emphasis is on theory, the number of problems to be solved is not great and it does not seem probable that the student will gain much facility in solving differential equations. Most of the techniques usually appearing in an introductory text are missing. In general, the exposition is clear but brief.

The book is pleasing in appearance and the number of typographical errors seems to have been kept to a minimum. The reviewer believes that most teachers of introductory courses in the subject would benefit by a careful reading of this text.

D. S. Morse
Union College

Differential Equations. By R. C. Yates. New York, McGraw-Hill Book Company, 1952. vii+215 pages. $3.75.

Most of the older text-books on Differential Equations approach the subject as follows: a general "type" differential equation is given, and then a "step one, step two, etc." process is exhibited which "solves" the equation. A course taught from such a book gives the student a great deal of drill in the manipulation of certain mathematical formulae, but teaches him very little in the way of understanding this subject which is of great importance in both pure and applied mathematics.

The author of the text under review proposes to remove one of the chief objections to such an approach by introducing each type through practical examples chosen from the fields of science. The preface states, "The structure of the text is built upon the realistic principle of having the student face the physical or geometrical situation first. He will thus recognize the need for a study of differential equations. He initially analyzes the problem and establishes the mathematical "set-up." Then his attention is directed to the selection and use of appropriate methods of solution. Finally, he must transcribe his mathematical results into the language of words. It is in this way that meaning and understanding are most firmly implanted."

The author has in general followed this procedure throughout the text, and the result is a text-book which is a definite improvement over the older method. The book contains many interesting and pertinent examples. Another interesting feature is the insertion of chapters on "Summary and Review Problems." It is believed that these will be helpful to both student and teacher.

The contents cover, in general, the material included in the usual introductory text-book on differential equations, the chapter titles being as follows: 1. Definitions; Formation of Equations; Physical and Geometrical Interpretations. 2. Separation of Variables and Homogeneity. 3. Equations of First Order, First Degree; The Bernoulli Equation; Integrable Forms. 4. Summary and Review Problems. 5. The Linear Equation with Constant Coefficients, Right Member Zero. 6. The Linear Equation of Second and Higher Order with Constant Coefficients, Right Member Not Zero; Equations with Variable Coefficients. 7. The Derivative Operator. 8. Summary and Review Problems. 9. Some Special Forms and Their Applications. 10. Approximate Numerical Solutions. 11. Summary and Review Problems. 12. Solutions in Series. 13. The Legendre and Bessel Equations; The Gamma Function. 14. Expansions. 15. Summary and Review Problems. 16. Engineering Problems Leading to Partial Differential Equations. 17. The Wave Equation and Separation of Variables. 18. Fourier Series. 19. Review and General Summary. Each chapter contains many well chosen problems of varying degree of difficulty.

For those who prefer a rather rigorous approach to the subject, it should be stated that the present text is written mainly for students of "applied mathematics." No existence theorems are stated—in fact no theorem is stated as such. The approach throughout is intuitive, and one who is an exponent of rigor would object to phrases such as, for example, those on page 22 which refer to dS as a small change in S, and dt as a small change in time. In most cases, no restrictions are placed on the functions discussed. For instance, in the chapter on Fourier Series, no conditions for convergence are stated.

In spite of the above criticism, the reviewer believes that this book will be welcomed by teachers of differential equations.

J. S. LEECH
Colorado College

Calculus of Variations. By Robert Weinstock. McGraw-Hill Book Company. 1950. x+326 pages. $6.50.

In the author's words, the two-fold purpose of this book is "(i) To provide for the senior or first-year graduate student in mathematics, science, or engineering an introduction to the ideas and techniques of the calculus of variations . . . (ii) To illustrate the application of the calculus of variations in several fields outside the realm of pure mathematics. (By far the greater emphasis is placed upon this second aspect of the book's purpose.)" Thus this book is primarily an introduction to calculus of variations for physicists and engineers; and it may well prove to be a popular one, for it appears to be generally well planned and carefully and clearly written. The only other book in English which is at all comparable in aim is C. Lanczos's *The Variational Principles of Mechanics*, University of Toronto Press (1949), a more traditional treatment concerned with presenting the role of calculus of variations in mechanics from the viewpoint of the history of ideas.

The desire to survey a variety of applications in a book of moderate size has forced upon the author some rather severe pruning of the material on calculus of variations proper. Thus not only are all questions of existence and sufficiency finessed, but there is no discussion whatever of second-order conditions or even of first-order corner conditions; problems in parametric form are barely mentioned; and for single integrals only the simplest variable-end point situations in the plane are treated. Problems with side conditions, being particularly important for applications, are considered somewhat more fully. Happily, sound one-parameter-embedding arguments replace use of the slippery "variational operator" δ. Less happily, the term "extremum" is used throughout as synonymous with "stationary value."

A list of the later chapter headings suggests the range of applications: geometrical optics: Fermat's principle; dynamics of particles; two independent variables: the vibrating string; the Sturm-Liouville eigenvalue-eigenfunction problem; several independent variables: the vibrating membrane; theory of elasticity; quantum mechanics; electrostatics. The treatment of several of these topics owes much to Courant-Hilbert. The Ritz method is illustrated early for the vibrating string and discussed in connection with the various other vibration problems. The chapter on quantum mechanics clearly sets forth the important role of variational methods in the early development of the subject; and one can well afford to be reminded that the variational approach to the hydrogen atom has definite virtues of elegance in comparison with the differential-equations method usual in current physics texts. There are numerous well-selected exercises some of which extend the theory developed in the text.

T. A. Botts
University of Virginia

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INSTITUTES FOR TEACHERS OF MATHEMATICS

The Association of Teachers of Mathematics in New England announces that its fifth annual Institute for Teachers of Mathematics will be held during August 20–27, 1953 at Colby College, Waterville, Maine.

The University of Houston announces its third Mathematics Institute to be held on its campus in Houston, Texas, June 23–26, 1953. For information write to C. B. Rader, Box 234, University of Houston, Houston 4, Texas.

The University of Washington will sponsor an Institute for Teachers of Mathematics to be held in Seattle, June 22–26, 1953. Further information may be obtained from the Department of Mathematics, University of Washington, Seattle 5, Washington.

A SUMMER CONFERENCE IN COLLEGIATE MATHEMATICS

There will be held at the University of Colorado in 1953 an eight-week Summer Conference in Collegiate Mathematics. The National Science Foundation first made the suggestion that such a conference be held, the Committee on the Regional Development of the National Research Council made the preliminary plans including designation of location and objectives, and the National Science Foundation has granted financial support.

The primary object of the conference will be to assist teachers in colleges and universities not closely associated with the big research centers to improve the quality of the undergraduate majors in mathematics; this to be accomplished by giving such teachers an opportunity to think and discuss mathematics under the guidance of men who are not only nationally known but who can give to non-specialists clear and inspiring pictures of the sweep and depths of present-day mathematics. Throughout the conference Professor Emil Artin of Princeton University will lecture on Modern Developments in Algebra, and Professor Wilder of the University of Michigan on Foundations of Analysis and Geometry. A number of additional lectures will be arranged, including a series by Professor G. Pólya on problem solving. Ample opportunity will be given for formal and informal discussion.

The members of the conference will be housed chiefly in one of the newest dormitories of the University where provision will be made for single persons, couples, and families at a cost of about $160 per person for room and board for the period of the conference—approximately June 15 to August 8. Some accommodations of other types will be available and the Director will be glad to

assist in securing other types of housing for those who desire it.

The plan includes stipends of $300 each for a limited number of participants and it is hoped that many can come without such support, in some cases with assistance from their own universities. Those who wish to participate are asked to apply for membership and agree to stay for the entire period of the conference. It is expected that such membership will be freely granted up to the limit of size imposed by accommodations available. Applications for housing must be made well in advance of the opening date. There will be no cost to participants in the conference except a one-dollar registration fee.

Recreational week-end opportunities in and about Boulder are many. It is about thirty-five miles from Rocky Mountain National Park and thirty miles from Denver. Red Rocks Amphitheater and the Central City summer opera are within an hour's drive. The University of Colorado has an extensive recreational program in the summer, including mountain hiking, bus trips, horseback riding, and other sports. One or two special excursions will probably be arranged for those attending the conference.

Notice of this conference has been sent to all heads of departments of mathematics and Presidents of colleges and universities listed in the directory of the Mathematical Association of America. For further information and application blanks for membership, stipends, and housing, write Burton W. Jones, Director, 119E Hellems Building, The University of Colorado, Boulder, Colorado. Applications for stipends must be in the hands of the director by April 1.

MARIE J. WEISS MEMORIAL SCHOLARSHIP FUND

A fund has been established to endow a scholarship at Newcomb College, Tulane University, in memory of Marie J. Weiss. Those desiring to give to this fund should mark their contributions "for the Marie J. Weiss Memorial Scholarship Fund," and mail them to the Tulane Alumni Fund, Tulane University, New Orleans, Louisiana.

PERSONAL ITEMS

Professor P. O. Bell of the University of Kansas has been awarded a National Science Foundation Postdoctoral Fellowship and is spending the year at the University of California, Berkeley.

Professor Claude Chevalley of Columbia University was elected to the National Academy of Sciences during 1952.

Mr. H. C. Griffith of the University of Tennessee has been awarded a National Science Foundation Fellowship for the academic year 1952–53.

Dr. John Wermer of Yale University has been awarded a National Science Foundation Fellowship for the year 1952–53.

Agricultural and Mechanical College of Texas announces: Associate Professors R. E. Basye and J. A. Daum have been promoted to professorships; Assistant Professor J. T. Kent has been promoted to an associate professorship; Dr. E. R. Keown has been appointed to an assistant professorship; Mr. D. B. Alexander has been appointed to an instructorship.

At Florida State University: Dr. J. W. Ellis, formerly a graduate student at Tulane University, Dr. M. J. Walsh, previously a graduate student at the University of Illinois, and Dr. R. E. Wheeler of the University of Kentucky, have been appointed to assistant professorships; Mr. C. M. Callahan, who has been an instructor at Lincoln Memorial University, has been appointed to an instructorship; Associate Professor Olga Larson has retired.

Lehigh University announces the following: Mr. Samuel Goldberg has been promoted to an assistant professorship; Dr. C. C. Hsiung, previously research fellow at Harvard University, has been appointed to an assistant professorship; Dr. S. I. Goldberg, who has been a research scientist with the Defense Research Board, Valcartier, Quebec, Canada, Dr. Felix Haas of Massachusetts Institute of Technology, Dr. J. A. Schatz, previously editorial assistant for Mathematical Reviews, and Dr. H. H. Wicke, formerly part-time instructor at State University of Iowa, have been appointed to instructorships; Mrs. Marjorie Halpern, Mr. R. R. Hohl, and Mr. C. B. Sensenig have been appointed to graduate assistantships; Associate Professor K. W. Lamson has retired with the title of Associate Professor Emeritus.

McMaster University makes the following announcements: Associate Professor J. D. Bankier has been appointed Chairman of the Department of Mathematics; Assistant Professor N. D. Lane of Carlton College, Ottawa, Canada, has been appointed to an assistant professorship.

Miami University reports the following: Associate Professor G. W. Spenceley has been promoted to a professorship; Mr. R. W. Emmert of Earlham College has been appointed to an instructorship.

Mississippi State College reports the following: Associate Professor S. B. Murray has been promoted to a professorship; Acting Instructor R. H. Hopkins has been promoted to an instructorship.

Washington Square College, New York University announces the following: Associate Professor Morris Kline has been promoted to a professorship; Dr. S. C. Lowell has been appointed Research Assistant Professor; Associate Professor W. M. Maiden and Professor R. G. Putnam have retired.

At Ohio University: Dr. Robert Butner of the University of Iowa has been appointed to an assistant professorship; Miss Mary Colberg, formerly associated with a research project of the University of Michigan, Mr. Neal Newby, Jr. of Harvard University, and Mr. Algray Verssen, who has been teaching in American schools in Germany, have been appointed to instructorships; Mrs. Neva Johnson has retired.

Pomona College makes the following announcements: Professor C. G. Jaeger, chairman of the Department of Mathematics, is on sabbatical leave during the second semester of 1952–53 and is in Europe; Mr. John Walsh has been appointed to a part-time instructorship.

Syracuse University announces: Associate Professor D. E. Kibbey has been promoted to a professorship; Assistant Professors Nancy Cole, H. W. Farnham, and R. D. Whitney have been promoted to associate professorships; Dr. O. O.

Pardee has been promoted to an assistant professorship; Dr. G. F. Leger, previously a mathematician at Bell Aircraft Company, has been appointed to an instructorship.

At Texas Southern University: Professor J. A. Pierce, acting head of the Department of Mathematics, has been appointed Chairman of Graduate School Administration; Mr. Calvin Culbreath, previously with the School of Vocational and Industrial Education of the University, has been appointed to an instructorship; Instructors A. H. Wardlaw and J. E. Westberry have been awarded General Education Board Fellowships and are on leave of absence.

University of Arizona announces the following appointments: Dr. G. M. Petersen of Brandon College to an instructorship; Mr. J. E. Householder and Mr. Gilbert Puente to assistantships.

University of Cincinnati makes the following announcements: Assistant Professor E. F. White has been promoted to an associate professorship; Professor C. A. Ludeke has been awarded a grant by the Research Corporation, New York City.

University of Delaware announces the following: Assistant Professor E. V. Lewis has been promoted to an associate professorship; Mr. G. O. Peters, previously research engineer at Franklin Institute, has been appointed to an assistant professorship; Assistant Instructor J. L. Howell of Yale University and Instructor W. L. Marshall of Hofstra College have been appointed to instructorships; Instructor A. C. Nelson is now a graduate student at the University of North Carolina.

At the University of Houston: Professor Albert Newhouse has been appointed Chairman of the Department of Mathematics; Mr. E. B. Adams, Mrs. Flora H. Bosworth, Mr. H. C. Lefkovits, and Mr. L. S. Lockingen have been serving as part-time instructors.

University of Kansas reports the following: Professor Robert Schatten is spending the year 1952–53 at the Institute for Advanced Study; Associate Professor J. L. Kelley of Tulane University has been appointed Visiting Associate Professor; Dr. W. F. Donoghue, Jr., formerly at the Applied Physics Laboratory, Silver Spring, Maryland, and Dr. K. T. Smith, who spent the year 1951–52 in Nancy, France, on a Fulbright Fellowship, have been appointed to assistant professorships; Dr. Arne Magnus, previously a graduate student at Washington University, has been appointed to an instructorship.

University of Kentucky announces: Associate Professor W. H. Pell of Brown University has been appointed Professor and Head of the Department of Mathematics; Professor H. H. Downing, formerly head of the Department of Mathematics, retains the position of Professor of Mathematics; Mr. D. C. Rose has been promoted to an instructorship; Graduate Assistants R. H. Sprague and W. C. Swift have been promoted to part-time instructorships; Mr. Ceslovas Masaitis of the University of Kaunas, Lithuania, has been appointed to an instructorship; Mr. Gene Adkins, Mr. Joseph Cornelison, Mr. T. F. Droege, Mr. M. I. Rose, and Mr. R. E. Shely have been appointed to graduate assistantships.

University of Maine announces the appointment of Dean Emeritus Paul Cloke of the College of Technology as Lecturer in the Department of Mathematics and the appointment of Mr. Phillip Hamm to a temporary instructorship.

University of Michigan announces the following: Associate Professor C. J. Nesbitt has been promoted to a professorship; Assistant Professor M. O. Reade has been promoted to an associate professorship; Instructors Raoul Bott, K. B. Leisenring, and A. J. Lohwater have been promoted to assistant professorships; Professor J. A. Dieudonné of the University of Nancy, France, has been appointed Visiting Professor for the year 1952–53; Dr. B. J. Tepping of the Bureau of Census has been appointed Lecturer; Dr. E. L. Griffin, formerly a graduate student at the University of Chicago, has been appointed to an instructorship; Assistant Professor N. H. Anning is on retirement furlough during 1952–53.

University of North Carolina at Chapel Hill reports the following: Dr. J. S. MacNerney of Northwestern University has been appointed to an assistant professorship; Professor T. F. Hickerson has retired with the title of Kenan Professor Emeritus.

At Woman's College, University of North Carolina: Assistant Professor Lila P. Walker has returned to the College after a year's leave of absence; Miss Frances Wolfe has been appointed Administrative Assistant to the Chancellor of the College.

University of Richmond announces the following: Assistant Professor D. F. Atkins of Bowling Green State University has been appointed to an assistant professorship; Dr. E. R. Sleight has retired from his position as Visiting Lecturer.

University of Saskatchewan makes the following announcements: Dr. G. H. M. Thomas, previously a graduate student at the University of Wisconsin, and Mr. Victor Linis, formerly a graduate student at McGill University, have been appointed to instructorships; Associate Professor Peter Scherk is on leave of absence and is Visiting Professor at University of California, Los Angeles.

University of South Carolina reports the following: Mr. Herbert Wolf, previously a graduate student at the University of North Carolina, has been appointed to an assistant professorship; Mrs. Elizabeth Wolf, who has been a graduate student at the University of North Carolina, has been appointed to an instructorship; Assistant Professor R. A. Lytle is on leave of absence and is studying at the University of Georgia.

University of Tennessee announces the following appointments to instructorships: Dr. Haskell Cohen, previously a graduate student at Tulane University; Mr. LaVerne Flatt, formerly a graduate student at George Peabody College for Teachers; Mrs. Ruth E. Hofstra of Syracuse University; Miss Thelma Peacock of the University of Maine.

University of Utah reports the following: Associate Professor C. J. Thorne has been promoted to a professorship; Professor R. N. Thomas is on leave of

absence for the academic year 1952–53 and is serving as a visiting professor at the Harvard College Observatory.

West Virginia University announces: Associate Professor J. K. Stewart has been promoted to a professorship; Assistant Professor A. B. Cunningham has been promoted to an associate professorship; Mr. R. A. Roberts, previously a teaching fellow at the University of Michigan, has been appointed to an assistant professorship; Mrs. Margaret S. Marshall has been appointed to an instructorship.

Yale University announces the following: Instructors R. R. Bernard and W. H. Mills have been promoted to assistant professorships; Dr. W. G. Bade of the University of California at Berkeley, Dr. R. G. Bartle, previously AEC Postdoctoral Fellow at Yale University, and Dr. Lawrence Markus, formerly research fellow and instructor at Harvard University, have been appointed to instructorships; also Dr. F. P. Pedersen, previously lecturer at the University of Southern California, Dr. F. D. Quigley, formerly teaching assistant at the University of Chicago, and Dr. J. T. Schwartz, who was an AEC Predoctoral Fellow at Princeton University, have been appointed to instructorships; Mr. P. C. Curtis, Jr., Mr. P. E. Klebe, Jr., Mr. T. A. Paley, and Mr. G. F. Simmons have been appointed to assistantships; Associate Professor J. I. Tracey has retired with the title of Associate Professor Emeritus.

Dr. J. W. Armstrong of Purdue University has accepted a position with the Aerophysics Section, Convair, Ft. Worth, Texas.

Associate Professor A. L. Blakers of Lehigh University has been appointed to a professorship at the University of Western Australia.

Mr. M. J. Bratt, formerly of the University of Cincinnati, is employed now by the General Electric Company, Cincinnati, Ohio.

Mr. H. W. F. Bruns, previously an instructor at Bowling Green State University, is teaching at Anthony Wayne High School, Waterville, Ohio.

Miss Helen Butcher, formerly an instructor at the University of Miami, is teaching at Miami Beach High School.

Dr. Peter Chiarulli of Carnegie Institute of Technology has been appointed to an assistant professorship in the Graduate Division of Applied Mathematics, Brown University.

Dr. G. F. Cramer of the Radio Corporation of America has a position as Staff Scientist with the Engineering Research Associates, Arlington, Virginia.

Dr. R. Y. Dean of California Institute of Technology has accepted a position as a mathematician with the General Electric Company, Richland, Washington.

Professor Chester Feldman of Antioch College has a position as a mathematical consultant with the Aircraft Marine Products, Harrisburg, Pennsylvania.

Mr. B. G. Fetterman, previously an instructor at Bowling Green State University, is teaching at Cleveland Heights High School, Ohio.

Dr. Mary E. Hamstrom of the University of Texas has been appointed to

an assistant professorship at Goucher College.

Assistant Professor L. A. Henkin of the University of Southern California has been promoted to an associate professorship.

Professor D. H. Hyers of the University of Southern California is on sabbatical leave during the current academic year and is at New York University.

Dr. R. V. Kadison, formerly at the Institute for Advanced Study, has been appointed to an assistant professorship at Columbia University.

Assistant Professor William Karush of the University of Chicago has been promoted to an associate professorship.

Professor Jacob Korevaar of the Institute of Technology, Delft, Netherlands, has been appointed to an assistant professorship at the University of Wisconsin.

Mr. R. E. Krucklin of the American Viscose Corporation has accepted a position as Senior Engineer with A. H. Johnson and Company, New York City.

Professor Hans Lewy has been appointed to a visiting professorship at Harvard University.

Mr. A. H. Moore of Pratt Institute has accepted a position as an operations analyst with the Operations Research Office, Johns Hopkins University.

Professor E. E. Moots of Cornell College has retired.

Dr. S. F. Neustadter of Harvard University is now a staff member with the Project Lincoln, Massachusetts Institute of Technology.

Mr. E. M. Olson has been appointed Lecturer at Columbia University.

Mr. R. E. Ozimkoski of Fordham University has been promoted to an assistant professorship.

Mr. P. E. Pfeiffer of Rice Institute has been promoted to the position of Assistant Professor of Electrical Engineering.

Dr. J. D. Riley, previously a graduate student at the University of Kansas, has a position as a mathematician with the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland.

Assistant Professor H. J. Ryser of Ohio State University has been promoted to an associate professorship.

Dr. H. S. Shapiro, formerly of Massachusetts Institute of Technology, is employed as a member of the technical staff of the Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. T. M. Simpson, Dean Emeritus of the Graduate School and Head Professor Emeritus of the Department of Mathematics, University of Florida, has accepted a position as Consultant in General Education for the Arkansas Experiment in Teacher Education sponsored by the Ford Foundation; he is located at Henderson State Teachers College.

Assistant Professor Wayman Strother of the University of Alabama has been appointed to an assistant professorship at the University of Miami.

Dr. J. A. Sullivan of the University of Notre Dame has been promoted to an assistant professorship.

Miss Helen J. Terry, previously an instructor at the University of Idaho, has a position with Sandia Corporation, Albuquerque, New Mexico.

Associate Professor F. E. Ulrich of Rice Institute has been promoted to a professorship.

Assistant Professor J. S. Vigder of the University of Saskatchewan has accepted a position with the Defence Research Board, Ottawa, Canada.

Dr. S. S. Walters has a position as an operations analyst with the Operations Research Office, Chevy Chase, Maryland.

Associate Professor L. E. Wear of California Institute of Technology has retired.

Mr. W. D. Wood, previously a graduate student at Purdue University, is now a staff member with the Sandia Corporation.

Mr. C. J. Kaufman died on November 10, 1952.

Dr. F. R. Moulton died on December 7, 1952. He was a charter member of the Association.

Professor Emeritus H. E. Van Buskirk of California Institute of Technology died on November 21, 1952. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 60 persons have been elected to membership by the Board of Governors on applications duly certified.

F. S. BADGER, M.S. (Western Reserve) Dean of Students, Alliance College.

B. J. BALL, Ph.D. (Texas) Acting Asst. Professor, University of Virginia.

R. D. BECKEY, B.A. (Wittenberg) Teacher and Director of Audio-Visual Education, Milton-Union High School, West Milton, Ohio.

C. J. BIGGERSTAFF, Electronics Technician, United States Navy.

A. J. BOSGANG, M.A. (C.C.N.Y.) Mathematician, Ballistics Research Laboratory, Aberdeen Proving Ground, Md.

J. R. BROWN, Ph.D. (Kansas) Asst. Professor, Bradley University.

H. S. CLAIR, M.S. (Chicago) Lecturer, Illinois Institute of Technology.

A. H. COCKSHOTT, M.A. (N.Y.U.) Asst. Professor, Manhattan College.

W. J. A. CULMER, M.A. (Minnesota) Instr., Departments of Mathematics and Mechanics, University of Minnesota.

E. K. DORFF, Student, University of Minnesota, Duluth, Minn.

M. S. DORGAN, M.S. (Iowa) Asst. Professor, Western State College, Gunnison, Colo.

E. I. DUGGAN, M.S. (Fordham) Instr., Iona College.

ERNA EANES, Student, Lenoir-Rhyne College.

J. R. ELLIOTT, B.S. (Alabama Poly.) 2nd Lieutenant, United States Marines.

T. J. ENGLERT, Student, Kent State University.
P. J. FINN, Student, St. John's College.
I. E. GLOVER, M.A. (Michigan) Asst. Professor, Langston University.
Mr. JEAN GREGOIRE, Student, Laval University.
J. B. HERRESHOFF, Student, University of California at Berkeley.
RUTH A. HUFFMAN, B.S. (Oklahoma) Statistician, Tinker Air Force Base, Oklahoma City, Okla.
REV. BROTHER CALIXTUS JAMES, M.A. (Manhattan) Head of Department, LaSalle Military Academy, Oakdale, N. Y.
W. D. JAMES, Student, Northern Illinois State Teachers College.
D. H. JONES, B.A. (Oberlin) Teaching Fellow, University of Washington.
R. P. KELISKY, B.S. (Texas Technological) Teaching Fellow, University of Texas.
H. W. KELLEY, Student, Alabama Polytechnic Institute.
R. R. KISSLING, 2611 Piedmont Avenue, Berkeley, Calif.
M. J. KNIEDLER, JR., M.S. (West Virginia) Engineer, Pratt and Whitney Aircraft, East Hartford, Conn.
L. H. LANGE, M.S. (Stanford) Instr., Valparaiso University.
J. S. LEW, Student, Yale University.
A. H. MCCABE, Student, St. John's College.
D. W. MCLEAN, B.S. (Wayne) Grad. Student, University of Chicago.
R. T. MCLEAN, M.A. (Bowling Green State) Instr., College of Steubenville.
E. B. MCLEOD, JR., M.S. (Stanford) Research Assistant, Stanford University.
R. E. MITCHELL, B.S. (Detroit) Gear Mathematician, Vinco Corporation, Detroit, Mich.
R. L. MOENTER, M.A. (Ohio State) Asso. Professor, Midland College.
MORRIS MONSKY, B.A. (Columbia) Associate Actuary, Mutual Life Insurance Company of New York, New York City.
J. E. MUELLER, Student, Butler University.
P. A. NURNBERGER, B.A. (Wabash) Head, Departments of Mathematics and Science, Tri-State College.
R. E. OZIMKOSKI, M.S. (Fordham) Asst. Professor, Fordham University.
E. C. PAXHIA, Student, University of Rochester.
K. C. PENG, M.S. (Michigan) Statistical Quality Control Engineer, Parke, Davis and Company, Detroit, Mich.
D. A. QUARLES, JR., M.A. (Yale) Senior Mathematician, Applied Science Department, International Business Machines Corporation, New York City.
D. M. Ross, M.S. (Northwestern) Substitute Teacher, City of Chicago High Schools.
A. A. SARDINAS, M.A. (Harvard) Associate Research Engineer, Burroughs Adding Machine Company, Philadelphia, Pa.
J. W. SEHESTEDT, B.D. (Southwestern Baptist Theological Seminary) Minister, Hoyt, Okla.
C. B. SENSENIG, B.S. (Franklin & Marshall) Grad. Assistant, Lehigh University.
Mrs. DOROTHY B. SHAFFER, M.A. (Radcliffe) Asst. Mathematician, Cornell Aeronautical Laboratory, Buffalo, N. Y.
E. S. SHAPIRO, M.A. (California) Mathematician, United States Naval Radiological Defense Laboratory, San Francisco, Calif.
M. G. SHULTS, M.S. (Math., Ft. Hays Kansas State; Meteor., C.I.T.) Instr., Northern Oklahoma Junior College, Tonkawa, Okla.
SISTER MARY STEPHANIE, Ph.D. (Catholic) Teacher, Georgian Court College.
PAUL SLEPIAN, B.S. (M.I.T.) Grad. Assistant, Brown University.
W. J. STEWART, Electrical Engineering Draftsman, San Antonio General Depot, Tex.
F. H. STILLINGER, JR., Student, University of Rochester.
MORRIS TENENBAUM, M.A. (Columbia) Grad. Student, Cornell University.
N. O. TIFFANY, M.A. (Southern California) Instr., Alfred University.
S. B. TOWNES, Ph.D. (Chicago) Asso. Professor, University of Hawaii.
MARION D. WETZEL, Ph.D. (Northwestern) Asso. Professor, Denison University.
H. H. WICKE, Ph.D. (Iowa) Instr., Lehigh University.
MRS. ELIZABETH S. WOLF, M.A. (North Carolina) Instr., University of South Carolina.
F. G. WOLFORT, JR., Student, St. John's College.

THE THIRTY-SIXTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-sixth annual meeting of the Mathematical Association of America was held at Washington University, Saint Louis, Missouri, on Tuesday, December 30, 1952, in conjunction with the annual meetings of the American Mathematical Society, the American Association for the Advancement of Science, and the Association for Symbolic Logic. About five hundred persons were registered, including the following two hundred and eighty-one members of the Association:

V. W. Adkisson, L. W. Akers, A. A. Albert, H. W. Alexander, C. B. Allendoerfer, R. D. Anderson, P. M. Anselone, Beulah Armstrong, Miriam C. Ayer, W. G. Bade, B. J. Ball, I. A. Barnett, M. A. Basoco, P. T. Bateman, J. D. Baum, P. R. Beesack, E. G. Begle, R. H. Bing, A. H. Black, H. D. Block, Brother J. A. Boose, R. D. Boswell, Jr., C. A. Bridger, R. W. Brink, R. H. Bruck, H. D. Brunk, P. B. Burcham, R. S. Burington, Jewell H. Bushey, R. K. Butz, S. S. Cairns, H. H. Campaigne, K. H. Carlson, J. O. Chellevold, Y. W. Chen, Elsie T. Church, R. V. Churchill, D. E. Coffey, L. A. Colquitt, Brother Damian Connelly, Byron Cosby, W. A. Couch, N. A. Court, Mary L. Cummings, J. C. Currie, Robert Davies, A. H. Diamond, Jim Douglas, Jr., T. L. Downs, W. C. Doyle, Nelson Dunford, W. L. Duren, William H. Durfee, J. M. Earl, J. C. Eaves, Samuel Eilenberg, R. E. Ekstrom, H. Margaret Elliott, J. G. Elliott, D. O. Ellis, Bernard Epstein, D. H. Erkiletian, Jr., H. J. Ettlinger, G. M. Ewing, W. H. Fagerstrom, F. D. Faulkner, H. M. Feldman, William Feller, W. T. Fishback, Harley Flanders, A. J. Flynn, W. C. Foreman, G. E. Forsythe, J. S. Frame, C. V. Fronabarger, H. M. Gehman, J. J. Gergen, Reverend F. J. Gerst, A. M. Gleason, H. E. Goheen, V. D. Gokhale, Michael Goldberg, W. A. Golomski, Reverend H. H. Gottbrath, S. H. Gould, R. F. Graesser, W. W. Graham, L. M. Graves, F. L. Griffin, J. S. Griffin, Jr., H. C. Griffith, H. T. Guard, D. L. Guy, Franklin Haimo, Edwin Halfar, A. E. Hallerberg, P. C. Hammer, H. W. Handsfield, J. R. Hanna, Frank Harary, H. G. Harp, Nola A. Haynes, P. W. Healy, I. L. Hebel, G. A. Hedlund, C. E. Heilman, Melvin Henriksen, I. N. Herstein, Fritz Herzog, Edwin Hewitt, T. H. Hildebrandt, J. J. L. Hinrichsen, F. E. Hohn, D. L. Holl, L. Aileen Hostinsky, Aughtum S. Howard, Reverend J. A. Hratz, S. T. Hu, G. B. Huff, Ralph Hull, M. Gweneth Humphreys, M. H. Ingraham, Fritz John, H. J. Johnson, L. W. Johnson, B. W. Jones, F. B. Jones, P. S. Jones, W. C. Kalinowski, Irving Kaplansky, J. L. Kelley, M. R. Kenner, J. R. F. Kent, D. E. Kibbey, V. L. Klee, Jr., S. C. Kleene, George Klein, L. A. Knowler, J. C. Koken, R. L. Lambert, C. E. Langenhop, R. E. Langer, Paolo Lanzano, Leo Lapidus, H. D. Larsen, W. G. Leavitt, A. S. Lee, Joseph Lehner, R. A. Leibler, Walter Leighton, B. L. Lercher, W. J. LeVeque, D. R. Lintvedt, H. D. Lipsich, Lee Lorch, T. A. Love, L. L. Lowenstein, C. I. Lubin, G. R. MacLane, Saunders MacLane, H. M. MacNeille, J. F. Manogue, C. G. Maple, Morris Marden, R. H. Marquis, J. M. Marr, Ella Marth, W. S. Massey, W. C. McDaniel, E. J. McShane, G. M. Merriman, R. J. Michel, R. R. Middlemiss, J. S. Minas, Nellie P. Miser, W. L. Miser, T. W. Moore, W. L. Moore, Thirza A. Mossman, H. T. Muhly, Zeev Nehari, Albert Newhouse, C. V. Newsom, T. A. Newton, C. O. Oakley, J. H. Oppenheim, Morris Ostrofsky, T. P. Palmer, S. T. Parker, G. W. Patterson, H. P. Pettit, C. F. Pinzka, C. G. Pitner, J. C. Polley, G. B. Price, J. E. Pryor, O. J. Ramler, J. F. Randolph, L. T. Ratner, L. M. Reagan, Mina Rees, R. F. Reeves, Francis Regan, Haim Reingold, J. G. Renno, P. R. Rider, J. K. Riess, J. D. Riley, R. F. Rinehart, L. A. Ringenberg, L. V. Robinson, L. D. Rodabaugh, I. H. Rose, Alex Rosenberg, P. C. Rosenbloom, J. B. Rosser, W. C. Royster, Charles Saltzer, Hans Samelson, R. G. Sanger, R. D. Schafer, Edith R. Schneckenburger, J. A. Schumaker, Nathan Schwid, L. L. Scott, W. R. Scott, R. R. Seeber, Jr., D. M. Seward, M. Anice Seybold, Harold Shniad, Marlow Sholander, Edward Silverman, Annette Sinclair, Sister M. Leonarda, Sister M. Pachomia, Sister Mary Teresine, Sister Presentation, Sister Rose Margaret, Sister Seraphim, M. F. Smiley, James C. Smith, R. L. Snider, W. S. Snyder, C. P. Sousley, E. J. Specht, R. L. Sternberg, A. G. Swanson, Gabor Szego, C. T. Taam, T. T. Tanimoto, Mildred E. Taylor, H. P. Thielman, R. H. Thompson, A. W. Tucker, J. L. Ullman, W. R. Utz,

B. O. Van Hook, G. B. Van Schaack, H. E. Vaughan, Reverend Dunstan Velesz, T. L. Wade, G. L. Walker, R. J. Walker, H. S. Wall, C. Y. Wang, J. A. Ward, L. E. Ward, Jr., A. M. Wedel, J. V. Wehausen, L. M. Weiner, Brother Bernard Alfred Welch, F. J. Weyl, P. M. Whitman, G. T. Whyburn, R. L. Wilder, W. L. Williams, J. L. Wilson, R. H. Wilson, Jr., G. N. Wollan, Alice K. Wright, J. W. T. Youngs.

Sessions of the Association were held on Tuesday morning and afternoon in Louderman Auditorium of Washington University. Vice-President Jewell H. Bushey presided at the morning session and President Saunders MacLane at the afternoon session. The Program Committee for the meeting consisted of Walter Leighton, Chairman, V. W. Adkisson, and M. K. Fort, Jr.

FIRST SESSION OF THE ASSOCIATION

"The Mathematical Work of Jacques Hadamard," by Professor Szolem Mandelbrojt, Rice Institute and College de France.

"Recent Applications of Convex Functions," by Professor J. W. Green, University of California at Los Angeles. (Presented by title due to illness of Professor Green.)

"The Scientific Research Program of the Office of Ordnance Research," by Dr. A. H. Diamond, Office of Ordnance Research, U. S. Army.

"Mathematics and the Educational Octopus," by Professor S. S. Cairns, University of Illinois.

SECOND SESSION OF THE ASSOCIATION

Symposium: The Teaching of Service Courses in Mathematics.

"Engineering Mathematics at Mid-Century," by Professor S. H. Caldwell, Electrical Engineering, Massachusetts Institute of Technology.

"What Shall We Teach in Service Courses?", by Professor F. E. Hohn, University of Illinois.

"Mathematics for Physicists, Pure or Applied?", by Professor J. W. Buchta, Physics, University of Minnesota.

"Abstract Mathematics for Scientists," by Professor W. L. Duren, Jr., Tulane University and National Science Foundation.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Monday morning in the Lounge of Brown Hall, with sixteen members present. Among the more important items of business transacted were the following:

Professor W. L. Duren, Jr., of Tulane University was elected Second Vice-President for 1953–1954.

Approval was given to the appointment by President MacLane of the following Nominating Committee for 1953: W. T. Martin, Chairman, J. C. Brixey, and G. K. Kalisch.

The Board voted to establish a standing committee on the Earle Raymond Hedrick Lectures and to discharge with thanks the Committee on Expository

Lectures (G. B. Price, Chairman, G. C. Evans, and J. C. Oxtoby) which had arranged for the first two series of Hedrick Lectures. It was also voted to approve the invitation to Professor P. R. Halmos of the University of Chicago to deliver the second series of Hedrick Lectures at the 1953 Summer Meeting on the subject of Axiomatic Set Theory.

The Thirty-fourth Summer Meeting of the Association will be held on August 31 and September 1, 1953, at Queen's University and the Royal Military College, Kingston, Ontario, Canada.

The Board voted to approve the appointment by the President of a committee to study the possible establishment of an employment bureau, of a joint committee (with the National Council of Teachers of Mathematics) on teacher education in mathematics, of a committee on the Undergraduate Mathematical Program, and of a joint committee (with the National Council of Teachers of Mathematics) to explore the possibility of publishing a mathematical journal for high school students. The President was also authorized to appoint two representatives on the United States sub-committee of the International Mathematical Instruction Committee.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Tuesday, December 30, 1952 at 2:00 P.M. in Louderman Auditorium of Washington University, Saint Louis, Missouri. President Saunders MacLane presided.

The Secretary announced the results of the balloting for officers, in which 1222 votes were cast. E. J. McShane of the University of Virginia was elected President for the two-year term 1953–1954. S. S. Cairns of the University of Illinois and A. W. Tucker of Princeton University were elected Governors for the three-year term 1953–1955.

MEETING OF SECTION OFFICERS

A meeting of Section Officers of the Association was held on Sunday evening in Room 107 of Brown Hall. President MacLane presided. Thirty-five persons were present representing twenty-one of the twenty-five sections of the Association. The following matters were discussed: problems pertaining to programs of section meetings, special activities of sections, such as contests for high school students, traveling lectureships, study of curriculum and teaching of secondary school mathematics, and collection of problem material from industry. Several suggestions for future activities were submitted. The method of electing sectional governors was described.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on Saturday, December 27, and continued through Monday, December 29. The Josiah Willard Gibbs Lecture was delivered by Professor Marston Morse of the Institute for Advanced Study on "Topology and Geometrical Analysis." The invited ad-

dress by Professor A. M. Gleason of Harvard University on "Natural Co-ordinate Systems" was awarded the Newcomb Cleveland Prize of one thousand dollars by a committee of the American Association for the Advancement of Science as a "noteworthy paper representing an outstanding contribution to science." Professor John von Neumann of the Institute for Advanced Study gave his Retiring Presidential Address on the topic: "A Logical Theory of Automata."

Section A of the American Association for the Advancement of Science held a single session on Monday at which Professor R. L. Wilder of the University of Michigan delivered his Retiring Vice-Presidential Address on "The Origin and Growth of Mathematical Concepts."

The Association for Symbolic Logic held its sessions on Monday morning and afternoon.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of R. R. Middlemiss, Chairman, T. L. Downs, H. Margaret Elliott, H. M. Gehman, Francis Regan, Marlow Sholander, and J. W. T. Youngs.

Registration headquarters was in the entry hall of the Women's Building of Washington University. Those attending the meetings were housed in hotels, including the Chase, Melbourne, Roosevelt and Sheraton Hotels. Meals were served in the cafeteria of Lee Hall.

Washington University entertained with coffee on Saturday evening in the Women's Building and tea was served on Sunday, Monday and Tuesday afternoons. A sightseeing tour of Saint Louis was held on Monday morning. An Opera Workshop Performance was presented on Sunday evening in the Auditorium of Brown Hall by the Music Department of Washington University.

A dinner for the members of the mathematical organizations and their guests was held on Monday at 6:30 P.M. in the Gymnasium of the Women's Building. Professor T. L. Downs of Washington University acted as toastmaster. Vice-Chancellor L. J. Buchan of Washington University brought greetings from the University and spoke about the scientific development of the University during the past century. Professor G. T. Whyburn, President-elect of the American Mathematical Society, spoke of the common interest of the organizations represented at the meeting in the advancement of mathematics. Professor L. M. Graves, Vice-President of Section A of the A.A.A.S., asked for the support of that organization. Professor C. B. Allendoerfer, representing the Mathematical Association of America as its Editor-in-chief, related the plans of the Association for carrying out its primary job of improving undergraduate instruction in mathematics. President J. B. Rosser of the Association for Symbolic Logic told of the successful meeting of that organization.

A motion prepared by Professor J. S. Frame was adopted by a rising vote of those present, expressing sincere thanks to the officers of Washington University for placing at our disposal the excellent facilities of the University and expressing

deep appreciation to the members of the local committee on arrangements whose cooperative efforts had made this meeting so pleasant and congenial.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held October 31, 1952 at Oklahoma City University, Oklahoma City, Oklahoma.

There were sixty-two in attendance, including the following thirty-seven members of the Association:

E. F. Allen, R. V. Andree, Arthur Bernhart, J. C. Bradford, J. C. Brixey, R. L. Caskey, N. A. Court, R. B. Deal, D. C. Dragoo, N. A. Eisen, I. E. Glover, A. A. Grau, E. V. Greer, L. A. Guest, O. H. Hamilton, J. O. Hassler, E. E. Heimann, J. E. Hoffman, W. N. Huff, P. W. M. John, L. W. Johnson, J. T. Krattiger, J. E. LaFon, Gene Levy, H. W. Linscheid, Dora McFarland, G. E. Meador, Mrs. Dorothea Meagher, R. R. Murphy, C. M. Pirrong, J. W. Sehestedt, H. W. Smith, C. E. Springer, Vivian Spurgeon, R. W. Veatch, G. R. Vick, J. H. Zant.

The following officers were elected for the coming year: Chairman, Professor W. N. Huff, University of Oklahoma; Vice-Chairman, Professor I. E. Glover, Langston University; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The following program was presented:

1. *Analytic functions with an irregular linear measurable set of singular points*, by Professor I. E. Glover, Langston University.

This paper will be published in the *Canadian Journal of Mathematics*.

2. *A mapping of the n-dimensional euclidean space on plane*, by Mr. A. Zirakzadeh, Oklahoma Agricultural and Mechanical College, introduced by the Chairman.

A set of n points $x_1, x_2, \cdots, x_n$ lying on n coplanar parallel lines $X_1, X_2, \cdots, X_n$ respectively, is called a point in the n-dimensional parallel space P^n. By a suitable choice of origins on these n parallel lines, it is possible to establish a one-to-one correspondence between points $P_e(a_1, \cdots, a_n)$ of an n-dimensional euclidean space E^n and points $(X_1, \cdots, X_n)$ of space P^n.

Two points $P_e(a_1, \cdots, a_n)$ and $P_e(b_1, \cdots, b_n)$ of the space E^n map into $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ of space P^n. The $n-1$ lines X_iX_{i+1} meet the $n-1$ lines Y_iY_{i+1} in $n-1$ points $x_{i-1,i}$. This set of $n-1$ points is the image of the line joining $P_e(a_1, \cdots, a_n)$ and $P_e(b_1, \cdots, b_n)$ in the space E^n. Any other point $P_e(c_1, \cdots, c_n)$ of this line maps into a point $(Z_1, \cdots, Z_n)$ such that the line Z_iZ_{i+1} passes through the point $X_{i-1,i}$.

This mapping is one-to-one and preserves incidence and parallelism. In case $n=2$ it reduces to a special form of duality in plane.

It is also possible to map planes and hyperplanes since they can be defined in terms of points and lines.

This mapping makes it possible to prove the theorems concerning incidence in plane or higher spaces with the use of only the axioms of the plane euclidean geometry. It also gives easier solutions to many of the problems concerning point and line.

An application of this mapping leads to a new graphical method for the solution of a system of 4 non-homogeneous linear equations in 4 unknowns.

3. *A generalization of the theorem of Meusnier*, by Professor C. E. Springer, University of Oklahoma.

Professor Springer developed a formula for the angle between the principal normal to a curve through a point P on a surface and the line through P of a rectilinear congruence. The formula reduces to that of the theorem of Meusnier in case the congruence is normal to the surface. A geometric interpretation for the general case was presented.

4. *Determination of the form of an empirical equation by numerical methods*, by Mr. N. A. Eisen, Tulsa, Oklahoma.

Divided differences and partial divided differences are defined and applied to polynomials, exponential functions, and power functions.

5. *The Riemann metric in plane affine geometry*, by Mr. R. B. Deal, University of Oklahoma.

The purpose is to point out the pedagogical advantages in using a Riemannian metric and linear affine transformations in an advanced plane analytics course where the student has a simple picture to guide his intuition as he is introduced to certain fundamental tensors and scalars.

6. *Some topological aspects of the four color problem*, by Professor O. H. Hamilton, Oklahoma Agricultural and Mechanical College.

A necessary and sufficient condition for the truth of the four color conjecture is given in terms of chains of sub-maps. The original and the dual maps are compared and some sufficient conditions not necessary and necessary conditions not sufficient for the truth of the four color proposition are given. In the class of minimal maps, a sufficient condition is that all vertices lie on a single simple closed curve. A necessary condition is that all vertices lie on the sum of a finite number of disjoint simple closed curves.

From a consideration of the duals of minimal maps, it follows that each triangulation of the sphere can be converted into a net of quadrilaterals on the sphere by deleting one and only one side from each triangle, if the four-color proposition is true.

7. *Three-dimensional plane geometry*, by Professor Arthur Bernhart, University of Oklahoma.

An exposition of the geometry of the triangle is achieved by constructing a three-dimensional figure and then projecting its points and lines onto the plane.

8. *On binary relations*, by Mr. C. J. Clark, Oklahoma Agricultural and Mechanical College, introduced by the Chairman.

A definition is given for what is called a C-R algebra. This abstract algebraic system, which is an extension of the notion of closure algebra, is a Boolean algebra with operators. In this system the closure operation is extended from elements of a Boolean algebra to those of a relation algebra. This necessitates additional axioms on operations and elements peculiar to relation algebras. Definitions are given for domain element, range element, image of certain types of elements under other elements, and theorems are established concerning these notions. The concept of continuity of a function is generalized to that of a relation, namely one that is closed, and theorems are established similar to those for continuous functions.

9. *Cryptographic systems*, by Professor R. V. Andree, University of Oklahoma.

This paper will appear in *Scripta Mathematica*, vol. XVIII, No. 1.

10. *On a "United States Copyright Angle Trisection,"* by Professor N. A. Court, University of Oklahoma.

The perennial problem of angle trisectors was discussed with special reference to a recent and persistent individual trisector whose copyright trisection many of the hearers have seen. It was suggested that if you yourself are not willing to solve the "find my error" puzzle, that you should *not* take the liberty of referring the trisector to someone else. Most teachers are capable of finding errors in such arguments if they are willing to take the required time. If they are not willing, they should not impose an identical burden on others. In the case of the particular trisector for instance, it is readily shown that his "trisection" would imply that, in a triangle in which the ratio of two sides is 2:1, the same ratio holds for the respectively opposite angles. This is not valid for a triangle whose angles are 90°, 60°, 30°.

The program was followed by a joint luncheon with the Mathematics teachers of the Oklahoma Education Association at which Professor F. E. Grossnickle of New Jersey State Teachers College spoke. Members of the Association were invited to attend a panel discussion for high-school mathematics teachers after the dinner.

R. V. ANDREE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May 2, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS, Washburn Municipal University of Topeka, April 11, 1953.

KENTUCKY, University of Louisville, May 9, 1953.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Proving Ground, Dahlgren, Virginia, May 2, 1953.

METROPOLITAN NEW YORK, Teachers College, Columbia University, March 28, 1953.

MICHIGAN, Wayne University, Detroit, April 18, 1953

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, April 24, 1953.

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PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

ROCKY MOUNTAIN, University of Colorado, Boulder, April 17–18, 1953.

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